Application of Intuitionistic Neutrosophic Graph Structures in Decision-Making

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Abstract

Fuzzy graph structures being an application of fuzzy sets to graph structures, are profusely applicable in social science and environmental science. A neutrosophic graph structures are more flexible and precise as compared to fuzzy graph structures. They are extensively helpful in decision-making problems. In this paper, we present concept of single-valued intuitionistic neutrosophic graph structures. We introduce the certain operations on single-valued intuitionistic neutrosophic graph structures and elaborate them with suitable examples. Further, we investigate some remarkable properties of these operators. Moreover, we discuss a highly worthwhile real-life application of single-valued intuitionistic neutrosophic graph structures in decision-making. Lastly, we elaborate general procedure of our application by designing an algorithm.

Keywords: Graph structure, Single-valued intuitionistic neutrosophic graph structure.
Mathematics Subject Classification 2010: 03E72, 05C72, 05C78, 05C99

1 Introduction

Graphical models are extensively useful tools for solving combinatorial problems of different fields including optimization, algebra, computer science, topology and operations research etc. Fuzzy graphical models are comparatively more close to nature, because in nature vagueness and ambiguity occurs. There are many complex phenomena and processes in science and technology having incomplete information. To deal such cases we needed a theory different from classical mathematics. Graph structures as generalized simple graphs are widely used for study of edge colored and edge signed graphs, also helpful and copiously used for studying large domains of computer science. Initially in 1965, Zadeh [25] proposed the notion of fuzzy sets to handle uncertainty in a lot of real applications. Fuzzy set theory is finding large number of applications in real time systems, where information inherent in systems has various levels of precision. Afterwards in 1986, Turksen [22] proposed the idea of interval-valued fuzzy set. But in various systems, there are membership and non-membership values, membership value is in favor of an event and non-membership value is against of that event. So, Atanassov [5] proposed the notion of intuitionistic fuzzy set in 1986. The intuitionistic fuzzy sets are more practical and applicable in real-life situations. Intuitionistic fuzzy set deal with incomplete information, that is, degree of membership function, non-membership function but not indeterminate and inconsistence information that exists definitely in many systems, including belief system, decision-support systems etc. That's why in 1998, Smarandache [19] proposed another notion of imprecise data named as neutrosophic sets. "Neutrosophic set is a part of neutrosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra". Neutrosophic set is recently proposed powerful formal framework. For convenient usage of neutrosophic sets in real-life situations, Wang et al. [23] proposed single-valued

neutrosophic(SVNS)set, that is generalized intuitionistic fuzzy set[5] with three independent components having values in unit interval [0, 1]. Euler initiated graph theory by solving seven bridges problem in 1973. In same year, Kauffman [13] defined fuzzy graph on the basis of Zadeh's fuzzy relations [26]. Rosenfeld [17] investigated fuzzy analogue of various graph-theoretic ideas in 1975. Later on, Bhattacharya gave some remarks on fuzzy graph in 1987. Bhutani and Rosenfeld discussed M-strong fuzzy graphs with their properties in [9]. Akram and Shahzadi [2] described the notion of neutrosophic soft graphs with applications. In 2011, Dinesh and Ramakrishnan [12] put forward fuzzy graph structures and investigated its properties. In 2016, Akram and Akmal [1] proposed the notion of bipolar fuzzy graph structures. In this paper, we present concept of single-valued intuitionistic neutrosophic graph structures. We introduce the certain operations on single-valued intuitionistic neutrosophic graph structures and elaborate them with suitable examples. Further, we investigate some remarkable properties of these operators. Moreover, we discuss a highly worthwhile real-life application of single-valued intuitionistic neutrosophic graph structures in decision-making. Lastly, we elaborate general procedure of our application by designing an algorithm.

2 Single-Valued Intuitionistic Neutrosophic Graph Structures

Definition 2.1. [7, 8] A set $\mathbf{I} = \{T_{\mathbf{I}}(v), I_{\mathbf{I}}(v), F_{\mathbf{I}}(v) : v \in U\}$ is said to be an *intuitionistic single-valued neutrosophic* (ISVNS)set if,

1. $\min\{T_{\mathbf{I}}(v), I_{\mathbf{I}}(v)\} \leq 0.5, \quad \min\{I_{\mathbf{I}}(v), F_{\mathbf{I}}(v)\} \leq 0.5, \quad \min\{F_{\mathbf{I}}(v), T_{\mathbf{I}}(v)\} \leq 0.5.$

2. $0 \leq T_{\mathbf{I}}(v) + I_{\mathbf{I}}(v) + F_{\mathbf{I}}(v) \leq 2.$

Definition 2.2. An ordered 3-tuple $G = (U, \mathbf{v}, \mathbf{e})$ is said to be a *single-valued intuitionistic neutrosophic graph*, where, $T_{\mathbf{v}}, F_{\mathbf{v}}, I_{\mathbf{v}} : U \to [0,1]$ denote truth, falsity and indeterminacy membership values of the vertices in U and $T_{\mathbf{e}}, F_{\mathbf{e}}, I_{\mathbf{e}} : U \times U \to [0,1]$ denote truth, falsity and indeterminacy membership values of the edges $(l, k) \in E \subseteq U \times U$ such that

1. $T_{\mathbf{e}}(l,k) \leq T(l) \wedge T(k), \quad F_{\mathbf{e}}(l,k) \leq F(l) \vee F(k), \quad I_{\mathbf{e}}(l,k) \leq I(l) \wedge I(k), \forall\, l, k \in U.$

2. $T_{\mathbf{e}}(l,k) \wedge I_{\mathbf{e}}(l,k) \leq 0.5, \quad T_{\mathbf{e}}(l,k) \wedge F_{\mathbf{e}}(k,l) \leq 0.5, \quad I_{\mathbf{e}}(l,k) \wedge F_{\mathbf{e}}(l,k) \leq 0.5,$

3. $0 \leq T_{\mathbf{e}}(l,k) + F_{\mathbf{e}}(l,k) + I_{\mathbf{e}}(l,k) \leq 2$

Definition 2.3. Consider a graph structure $\check{G} = (P, P_1, P_2, \ldots, P_r)$, then $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ is said to be a *single-valued intuitionistic neutrosophic graph structure(SVINSGS)*, if $O = \,< l, T(l), I(l), F(l) >$ and $O_h = \,< (l, k), T_h(l, k), I_h(l, k), F_h(l, k) >$ are single-valued intuitionistic neutrosophic(SVINS) sets on the sets P and P_h respectively such that

1. $T_h(l,k) \leq T(l) \wedge T(k), \quad I_h(l,k) \leq I(l) \wedge I(k), \quad F_h(l,k) \leq F(l) \vee F(k),$

2. $T_h(l,k) \wedge I_h(l,k) \leq 0.5, \quad T_h(l,k) \wedge F_h(l,k) \leq 0.5, \quad I_h(l,k) \wedge F_h(l,k) \leq 0.5,$

3. $0 \leq T_h(l,k) + I_h(l,k) + F_h(l,k) \leq 2, \quad for\ all\ (l, k) \in O_h$ where $h \in \{1, 2, \ldots, r\}.$

Where, O and O_h are underlying vertex and h-edge sets of SVINSGS $\check{G}_i$ $(h \in \{1, 2, \ldots, r\})$.

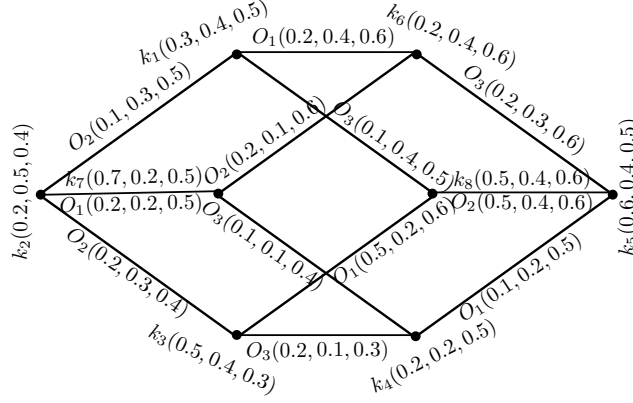**Example 2.4.** A single-valued intuitionistic graph structure is represented in Fig. 2.1.

Figure 2.1: A single-valued intuitionistic neutrosophic graph structure

Now we define the operations on SVINSGSs.

Definition 2.5. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, \ldots, Q_{2r})$ be SVINSGSs. *Cartesian product* of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \times \check{G}_{i2} = (O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}, \ldots, O_{1r} \times O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1 \times O_2)}(kl) = (T_{O_1} \times T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \times O_2)}(kl) = (I_{O_1} \times I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \times O_2)}(kl) = (F_{O_1} \times F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $(kl) \in P_1 \times P_2$,

(ii) $\begin{cases} T_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \times T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1 l_2) \\ I_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \times I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1 l_2) \\ F_{(O_{1h} \times O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \times F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1 l_2) \end{cases}$
for all $k \in P_1$, $(l_1 l_2) \in P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h} \times O_{2h})}(k_1 l)(k_2 l) = (T_{O_{1h}} \times T_{O_{2h}})(k_1 l)(k_2 l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1 k_2) \\ I_{(O_{1h} \times O_{2h})}(k_1 l)(k_2 l) = (I_{O_{1h}} \times I_{O_{2h}})(k_1 l)(k_2 l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1 k_2) \\ F_{(O_{1h} \times O_{2h})}(k_1 l)(q_2 l) = (F_{O_{1h}} \times F_{O_{2h}})(k_1 l)(k_2 l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1 k_2) \end{cases}$
for all $l \in P_2$, $(k_1 k_2) \in P_{1h}$.

Example 2.6. Consider $\check{G}_{i1} = (O_1, O_{11}, O_{12})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22})$ are SVINSGSs of GRSs $\check{G}_1 = (P_1, P_{11}, P_{12})$ and $\check{G}_2 = (P_2, P_{21}, P_{22})$ respectively, as represented in Fig. 2.2, where $P_{11} = \{k_1 k_2\}$, $P_{12} = \{k_3 k_4\}$, $P_{21} = \{l_1 l_2\}$, $P_{22} = \{l_2 l_3\}$.
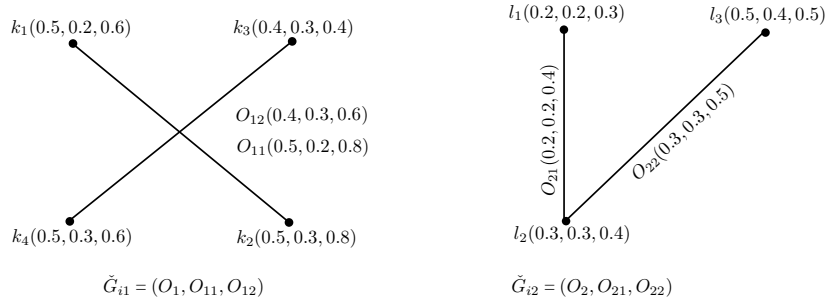

Figure 2.2: Two SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$

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Cartesian product of $\breve{G}_{i1}$ and $\breve{G}_{i2}$ defined as $\breve{G}_{i1} \times \breve{G}_{i2} = \{O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}\}$ is represented in Fig. 2.3.
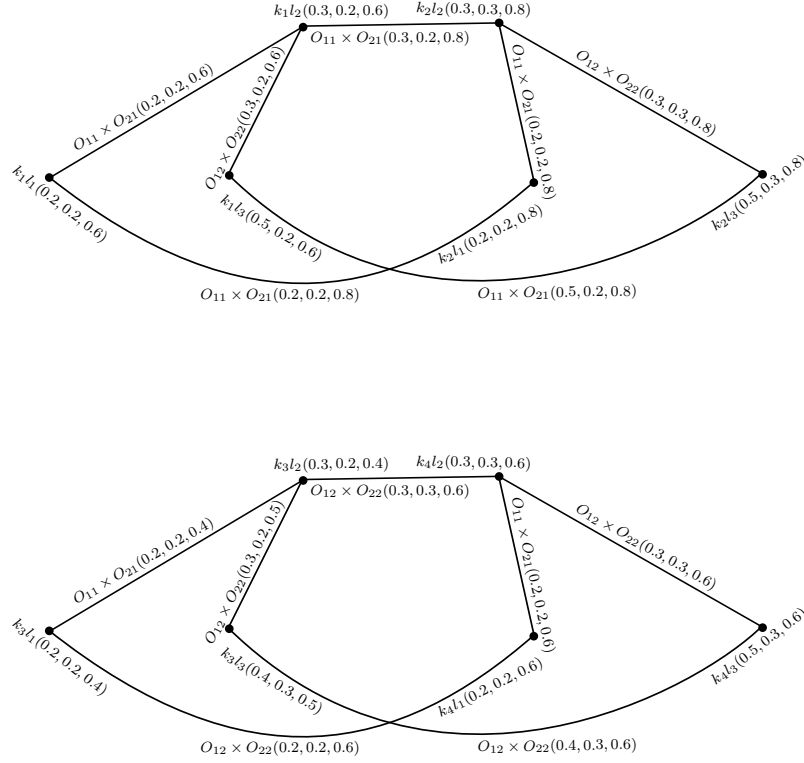

Figure 2.3: $\breve{G}_{i1} \times \breve{G}_{i2}$

Theorem 2.7. *Cartesian product* $\breve{G}_{i1} \times \breve{G}_{i2} = (O_1 \times O_2, O_{11} \times O_{21}, O_{12} \times O_{22}, \ldots, O_{1r} \times O_{2r})$ *of two SVINSGSs of GRSs $\breve{G}_1$ and $\breve{G}_2$ is a SVINSGS of $\breve{G}_1 \times \breve{G}_2$.*

Proof. We consider two cases:

Case 1. For $k \in P_1$, $l_1 l_2 \in P_{2h}$

$$
\begin{aligned}
T_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(r_1 l_2) \\
&\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\
&= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\
&= T_{(O_1 \times O_2)}(kl_1) \wedge T_{(O_1 \times O_2)}(kl_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1 l_2) \\
&\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\
&= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\
&= I_{(O_1 \times O_2)}(kl_1) \wedge I_{(O_1 \times O_2)}(kl_2),
\end{aligned}
$$

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$$F_{(O_{1h} \times O_{2h})}((kl_1)(kl_2)) = F_{O_1}(k) \vee F_{O_{2h}}(l_1 l_2)$$
$$\leq F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)]$$
$$= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)]$$
$$= F_{(O_1 \times O_2)}(kl_1) \vee F_{(O_1 \times O_2)}(kl_2),$$

for $kl_1, kl_2 \in P_1 \times P_2$.

Case 2. For $k \in P_2$, $l_1 l_2 \in P_{1h}$

$$T_{(O_{1h} \times O_{2h})}((l_1 k)(l_2 k)) = T_{O_2}(k) \wedge T_{O_{1h}}(l_1 l_2)$$
$$\leq T_{O_2}(k) \wedge [T_{O_1}(l_1) \wedge T_{O_1}(l_2)]$$
$$= [T_{O_2}(k) \wedge T_{O_1}(l_1)] \wedge [T_{O_2}(k) \wedge T_{O_1}(l_2)]$$
$$= T_{(O_1 \times O_2)}(l_1 k) \wedge T_{(O_1 \times O_2)}(l_2 k),$$

$$I_{(O_{1h} \times O_{2h})}((l_1 k)(l_2 k)) = I_{O_2}(k) \wedge I_{O_{1h}}(l_1 l_2)$$
$$\leq I_{O_2}(k) \wedge [I_{O_1}(l_1) \wedge I_{O_1}(l_2)]$$
$$= [I_{O_2}(k) \wedge I_{O_1}(l_1)] \wedge [I_{O_2}(k) \wedge I_{O_1}(l_2)]$$
$$= I_{(O_1 \times O_2)}(l_1 k) \wedge I_{(O_1 \times O_2)}(l_2 k),$$

$$F_{(O_{1h} \times O_{2h})}((l_1 k)(l_2 k)) = F_{O_2}(k) \vee F_{O_{1h}}(l_1 l_2)$$
$$\leq F_{O_2}(k) \vee [F_{O_1}(l_1) \vee F_{O_1}(l_2)]$$
$$= [F_{O_2}(k) \vee F_{O_1}(l_1)] \vee [F_{O_2}(k) \vee F_{O_1}(l_2)]$$
$$= F_{(O_1 \times O_2)}(l_1 k) \vee F_{(O_1 \times O_2)}(l_2 k),$$

for $l_1 k, l_2 k \in P_1 \times P_2$.

Both cases exists $\forall h \in \{1, 2, \ldots, r\}$. Hence we get proof. $\qquad\square$

Definition 2.8. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, Q_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, \ldots, Q_{2r})$ be SVINSGSs. *Cross product* of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} * \check{G}_{i2} = (O_1 * O_2, O_{11} * O_{21}, O_{12} * O_{22}, \ldots, O_{1r} * O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1 * O_2)}(kl) = (T_{O_1} * T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 * O_2)}(kl) = (I_{O_1} * I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 * O_2)}(kl) = (F_{O_1} * F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $(kl) \in P_1 \times P_2$,

(ii) $\begin{cases} T_{(O_{1h} * O_{2h})}(k_1 l_1)(k_2 l_2) = (T_{O_{1h}} * T_{O_{2h}})(k_1 l_1)(k_2 l_2) = T_{O_{1h}}(k_1 k_2) \wedge T_{O_{2h}}(l_1 l_2) \\ I_{(O_{1h} * O_{2h})}(k_1 l_1)(k_2 l_2) = (I_{O_{1h}} * I_{O_{2h}})(k_1 l_1)(k_2 l_2) = I_{O_{1h}}(k_1 k_2) \wedge I_{O_{2h}}(l_1 l_2) \\ F_{(O_{1h} * O_{2h})}(k_1 l_1)(k_2 l_2) = (F_{O_{1h}} * F_{O_{2h}})(k_1 l_1)(k_2 l_2) = F_{O_{1h}}(k_1 k_2) \vee F_{O_{2h}}(l_1 l_2) \end{cases}$
for all $(k_1 k_2) \in P_{1h}$, $(l_1 l_2) \in P_{2h}$.

Example 2.9. Cross product of SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2.2 is defined as $\check{G}_{i1} * \check{G}_{i2} = \{O_1 * O_2, O_{11} * O_{21}, O_{12} * O_{22}\}$ and is represented in Fig. 2.4.
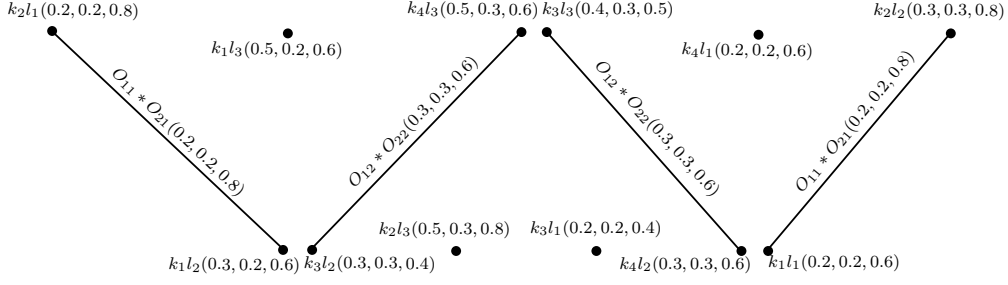
$k_2l_1(0.2,0.2,0.8)$ $k_1l_3(0.5,0.2,0.6)$ $k_4l_3(0.5,0.3,0.6)$ $k_3l_3(0.4,0.3,0.5)$ $k_4l_1(0.2,0.2,0.6)$ $k_2l_2(0.3,0.3,0.8)$

$O_{11}*O_{21}(0.2,0.2,0.8)$ $O_{12}*O_{22}(0.3,0.3,0.6)$ $O_{12}*O_{22}(0.3,0.3,0.6)$ $O_{11}*O_{21}(0.2,0.2,0.8)$

$k_2l_3(0.5,0.3,0.8)$ $k_3l_1(0.2,0.2,0.4)$

$k_1l_2(0.3,0.2,0.6)$ $k_3l_2(0.3,0.3,0.4)$ $k_4l_2(0.3,0.3,0.6)$ $k_1l_1(0.2,0.2,0.6)$

Figure 2.4: $\check{G}_{i1} * \check{G}_{i2}$

Theorem 2.10. *Cross product $\check{G}_{i1}*\check{G}_{i2} = (O_1*O_2, O_{11}*O_{21}, O_{12}*O_{22}, \ldots, O_{1r}*O_{2r})$ of two SVINSGSs of GRSs $\check{G}_1$ and $\check{G}_2$ is a SVINSGS of $\check{G}_1 * \check{G}_2$.*

Proof. For all $k_1l_1, k_2l_2 \in P_1 * P_2$

$$\begin{aligned} T_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k_2) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1*O_2)}(k_1l_1) \wedge T_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}l_1l_2) \\ &\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k_2) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1*O_2)}(k_1l_1) \wedge I_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

$$\begin{aligned} F_{(O_{1h}*O_{2h})}((k_1l_1)(k_2l_2)) &= F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \\ &\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\ &= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k_2) \vee F_{O_2}(l_2)] \\ &= F_{(O_1*O_2)}(k_1l_1) \vee F_{(O_1*O_2)}(k_2l_2), \end{aligned}$$

for $h \in \{1, 2, \ldots, r\}$. This gives proof. $\qquad\square$

Definition 2.11. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be SVINSGSs. *Lexicographic product* of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \bullet \check{G}_{i2} = (O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}, \ldots, O_{1r} \bullet O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1 \bullet O_2)}(kl) = (T_{O_1} \bullet T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \bullet O_2)}(kl) = (I_{O_1} \bullet I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \bullet O_2)}(kl) = (F_{O_1} \bullet F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $(kl) \in P_1 \times P_2$

(ii) $\begin{cases} T_{(O_{1h} \bullet O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \bullet T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h} \bullet O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \bullet I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h} \bullet O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \bullet F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$
for all $k \in P_1$, $(l_1l_2) \in P_{2h}$,

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$$\text{(iii)} \begin{cases} T_{(O_{1h}\bullet O_{2h})}(k_1l_1)(k_2l_2) = (T_{O_{1h}}\bullet T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h}\bullet O_{2h})}(k_1l_1)(k_2l_2) = (I_{O_{1h}}\bullet I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h}\bullet O_{2h})}(k_1l_1)(k_2l_2) = (F_{O_{1h}}\bullet F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2) \end{cases}$$

for all $(k_1k_2) \in P_{1h}$, $(l_1l_2) \in P_{2h}$.

Example 2.12. *Lexicographic product* of SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2.2 is defined as $\check{G}_{i1} \bullet \check{G}_{i2} = \{O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}\}$ and is represented in Fig. 2.5.
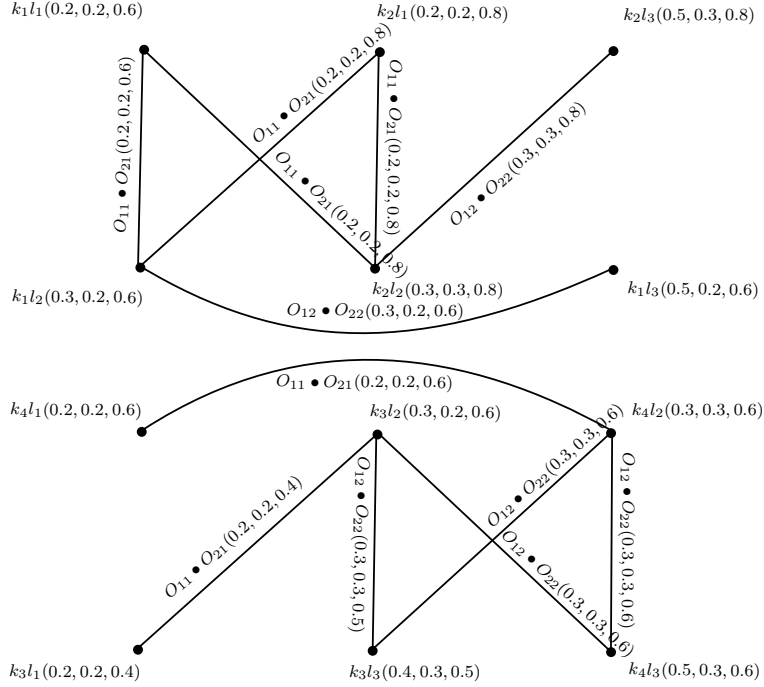

Figure 2.5: $\check{G}_{i1} \bullet \check{G}_{i2}$

Theorem 2.13. *Lexicographic product* $\check{G}_{i1} \bullet \check{G}_{i2} = (O_1 \bullet O_2, O_{11} \bullet O_{21}, O_{12} \bullet O_{22}, \ldots, O_{1r} \bullet O_{2r})$ *of two SVINSGSs of the GRSs $\check{G}_1$ and $\check{G}_2$ is a SVINSGS of $\check{G}_1 \bullet \check{G}_2$.*

Proof. We consider two cases:

Case 1. For $k \in P_1$, $l_1l_2 \in P_{2h}$

$$\begin{aligned} T_{(O_{1h}\bullet O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ &\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1\bullet O_2)}(kl_1) \wedge T_{(O_1\bullet O_2)}(kl_2), \end{aligned}$$

$$\begin{aligned} I_{(O_{1h}\bullet O_{2i})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ &\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\ &= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\ &= I_{(O_1\bullet O_2)}(kl_1) \wedge I_{(O_1\bullet O_2)}(kl_2), \end{aligned}$$

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$$F_{(O_{1h} \bullet O_{2i})}((kl_1)(kl_2)) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2)$$
$$\leq F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)]$$
$$= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)]$$
$$= F_{(O_1 \bullet O_2)}(kl_1) \vee F_{(O_1 \bullet O_2)}(kl_2),$$

for $kl_1, kl_2 \in P_1 \bullet P_2$.

Case 2. For $k_1k_2 \in P_{1h}, l_1l_2 \in P_{2h}$

$$T_{(O_{1h} \bullet O_{2h})}((k_1l_1)(k_2l_2)) = T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2)$$
$$\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)]$$
$$= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k_2) \wedge T_{O_2}(l_2)]$$
$$= T_{(O_1 \bullet O_2)}(k_1l_1) \wedge T_{(O_1 \bullet O_2)}(k_2l_2),$$

$$I_{(O_{1h} \bullet O_{2h})}((k_1l_1)(k_2l_2)) = I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2)$$
$$\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)]$$
$$= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k_2) \wedge I_{O_2}(l_2)]$$
$$= I_{(O_1 \bullet O_2)}(k_1l_1) \wedge I_{(O_1 \bullet O_2)}(k_2l_2),$$

$$F_{(O_{1h} \bullet O_{2h})}((k_1l_1)(k_2l_2)) = F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2)$$
$$\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)]$$
$$= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k_2) \vee F_{O_2}(l_2)]$$
$$= F_{(O_1 \bullet O_2)}(k_1l_1) \vee F_{(O_1 \bullet O_2)}(k_2l_2),$$

for $k_1l_1, k_2l_2 \in P_1 \bullet P_2$.
Both cases hold for $h \in \{1, 2, \dots, r\}$. Hence we get proof.

$\square$

Definition 2.14. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \dots, O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, \dots, O_{2r})$ be SVINSGSs. *Strong product* of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \boxtimes \check{G}_{i2} = (O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}, \dots, O_{1r} \boxtimes O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1 \boxtimes O_2)}(kl) = (T_{O_1} \boxtimes T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \boxtimes O_2)}(kl) = (I_{O_1} \boxtimes I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \boxtimes O_2)}(kl) = (F_{O_1} \boxtimes F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $(kl) \in P_1 \times P_2$,

(ii) $\begin{cases} T_{(O_{1h} \boxtimes O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h} \boxtimes O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h} \boxtimes O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$
for all $k \in P_1$, $(l_1l_2) \in P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h} \boxtimes O_{2h})}(k_1l)(k_2l) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1l)(k_2l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1k_2) \\ I_{(O_{1h} \boxtimes O_{2h})}(k_1l)(k_2l) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1l)(k_2l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1k_2) \\ F_{(O_{1h} \boxtimes O_{2h})}(k_1l)(k_2l) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1l)(k_2l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1k_2) \end{cases}$
for all $l \in P_2$, $(k_1k_2) \in P_{1h}$,

$$(iv) \begin{cases} T_{(O_{1h} \boxtimes O_{2h})}(k_1 l_1)(k_2 l_2) = (T_{O_{1h}} \boxtimes T_{O_{2h}})(k_1 l_1)(k_2 l_2) = T_{O_{1h}}(k_1 k_2) \wedge T_{O_{2h}}(l_1 l_2) \\ I_{(O_{1h} \boxtimes O_{2h})}(k_1 l_1)(k_2 l_2) = (I_{O_{1h}} \boxtimes I_{O_{2h}})(k_1 l_1)(k_2 l_2) = I_{O_{1h}}(k_1 k_2) \wedge I_{O_{2h}}(l_1 l_2) \\ F_{(O_{1h} \boxtimes O_{2h})}(k_1 l_1)(k_2 l_2) = (F_{O_{1h}} \boxtimes F_{O_{2h}})(k_1 l_1)(k_2 l_2) = F_{O_{1h}}(k_1 k_2) \vee F_{O_{2h}}(l_1 l_2) \end{cases}$$
for all $(k_1 k_2) \in P_{1h}$, $(l_1 l_2) \in P_{2h}$.

Example 2.15. Strong product SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2.2 is defined as $\check{G}_{i1} \boxtimes \check{G}_{i2} = \{O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}\}$ and is represented in Fig. 2.6.
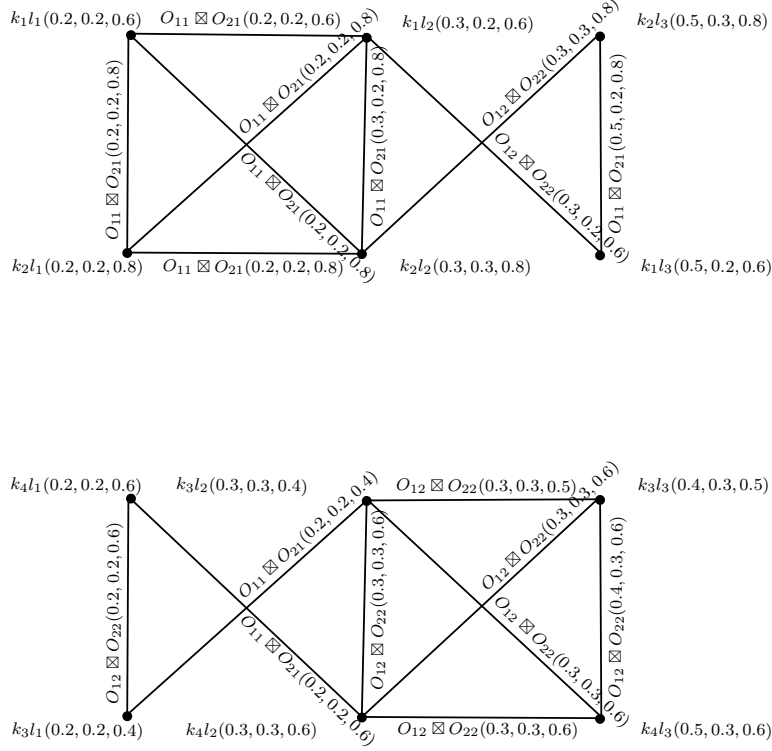

Figure 2.6: $\check{G}_{i1} \boxtimes \check{G}_{i2}$

Theorem 2.16. *Strong product* $\check{G}_{i1} \boxtimes \check{G}_{i2} = (O_1 \boxtimes O_2, O_{11} \boxtimes O_{21}, O_{12} \boxtimes O_{22}, \ldots, O_{1r} \boxtimes O_{2r})$ *of two SVINSGSs of the GRSs* $\check{G}_1$ *and* $\check{G}_2$ *is SVINSGS of* $\check{G}_1 \boxtimes \check{G}_2$.

Proof. There are three cases:

Case 1. For $k \in P_1$, $l_1 l_2 \in P_{2h}$

$$\begin{aligned} T_{(O_{1h} \boxtimes O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1 l_2) \\ &\leq T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\ &= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\ &= T_{(O_1 \boxtimes O_2)}(kl_1) \wedge T_{(O_1 \boxtimes O_2)}(kl_2), \end{aligned}$$

$$I_{(O_{1h}\boxtimes O_{2h})}((kl_1)(kl_2)) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2)$$
$$\leq I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)]$$
$$= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)]$$
$$= I_{(O_1\boxtimes O_2)}(kl_1) \wedge I_{(O_1\boxtimes O_2)}(kl_2),$$

$$F_{(O_{1h}\boxtimes O_{2h})}((kl_1)(kl_2)) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2)$$
$$\leq F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)]$$
$$= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)]$$
$$= F_{(O_1\boxtimes O_2)}(kl_1) \vee F_{(O_1\boxtimes O_2)}(kl_2),$$

for $kl_1, kl_2 \in P_1 \boxtimes P_2$.

Case 2. For $k \in P_2$, $l_1l_2 \in P_{1h}$

$$T_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) = T_{O_2}(k) \wedge T_{O_{1h}}(l_1l_2)$$
$$\leq T_{O_2}(k) \wedge [T_{O_1}(l_1) \wedge T_{O_1}(l_2)]$$
$$= [T_{O_2}(k) \wedge T_{O_1}(l_1)] \wedge [T_{O_2}(k) \wedge T_{O_1}(l_2)]$$
$$= T_{(O_1\boxtimes O_2)}(l_1k) \wedge T_{(O_1\boxtimes O_2)}(l_2k),$$

$$I_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) = I_{O_2}(k) \wedge I_{O_{1h}}(l_1l_2)$$
$$\leq I_{O_2}(k) \wedge [I_{O_1}(l_1) \wedge I_{O_1}(l_2)]$$
$$= [I_{O_2}(k) \wedge I_{O_1}(l_1)] \wedge [I_{O_2}(k) \wedge I_{O_1}(l_2)]$$
$$= I_{(O_1\boxtimes O_2)}(l_1k) \wedge I_{(O_1\boxtimes O_2)}(l_2k),$$

$$F_{(O_{1h}\boxtimes O_{2h})}((l_1k)(l_2k)) = F_{O_2}(k) \vee F_{O_{1h}}(l_1l_2)$$
$$\leq F_{O_2}(k) \vee [F_{O_1}(l_1) \vee F_{O_1}(l_2)]$$
$$= [F_{O_2}(k) \vee F_{O_1}(l_1)] \vee [F_{O_2}(k) \vee F_{O_1}(l_2)]$$
$$= F_{(O_1\boxtimes O_2)}(l_1k) \vee F_{(O_1\boxtimes O_2)}(l_2k),$$

for $l_1k, l_2k \in P_1 \boxtimes P_2$.

Case 3. For every $k_1k_2 \in P_{1h}, l_1l_2 \in P_{2h}$

$$T_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) = T_{O_{1h}}(k_1k_2) \wedge T_{O_{2h}}(l_1l_2)$$
$$\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)]$$
$$= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k_2) \wedge T_{O_2}(l_2)]$$
$$= T_{(O_1\boxtimes O_2)}(k_1l_1) \wedge T_{(O_1\boxtimes O_2)}(k_2l_2),$$

$$I_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) = I_{O_{1h}}(k_1k_2) \wedge I_{O_{2h}}(l_1l_2)$$
$$\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)]$$
$$= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k_2) \wedge I_{O_2}(l_2)]$$
$$= I_{(O_1\boxtimes O_2)}(k_1l_1) \wedge I_{(O_1\boxtimes O_2)}(k_2l_2),$$

$$F_{(O_{1h}\boxtimes O_{2h})}((k_1l_1)(k_2l_2)) = F_{O_{1h}}(k_1k_2) \vee F_{O_{2h}}(l_1l_2)$$
$$\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)]$$
$$= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k_2) \vee F_{O_2}(l_2)]$$
$$= F_{(O_1\boxtimes O_2)}(k_1l_1) \vee F_{(O_1\boxtimes O_2)}(k_2l_2),$$

for $k_1l_1, k_2l_2 \in P_1 \boxtimes P_2$, and $h = 1, 2, \ldots, r$.

This gives proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 2.17. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be SVINSGSs. *Composition* of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \circ \check{G}_{i2} = (O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}, \ldots, O_{1r} \circ O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1 \circ O_2)}(kl) = (T_{O_1} \circ T_{O_2})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_1 \circ O_2)}(kl) = (I_{O_1} \circ I_{O_2})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_1 \circ O_2)}(kl) = (F_{O_1} \circ F_{O_2})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $(kl) \in P_1 \times P_2$,

(ii) $\begin{cases} T_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (T_{O_{1h}} \circ T_{O_{2h}})(kl_1)(kl_2) = T_{O_1}(k) \wedge T_{O_{2h}}(l_1l_2) \\ I_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (I_{O_{1h}} \circ I_{O_{2h}})(kl_1)(kl_2) = I_{O_1}(k) \wedge I_{O_{2h}}(l_1l_2) \\ F_{(O_{1h} \circ O_{2h})}(kl_1)(kl_2) = (F_{O_{1h}} \circ F_{O_{2h}})(kl_1)(kl_2) = F_{O_1}(k) \vee F_{O_{2h}}(l_1l_2) \end{cases}$
for all $k \in P_1$, $(l_1l_2) \in P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h} \circ O_{2h})}(k_1l)(k_2l) = (T_{O_{1h}} \circ T_{O_{2h}})(k_1l)(k_2l) = T_{O_2}(l) \wedge T_{O_{1h}}(k_1k_2) \\ I_{(O_{1h} \circ O_{2h})}(k_1l)(k_2l) = (I_{O_{1h}} \circ I_{O_{2h}})(k_1l)(k_2l) = I_{O_2}(l) \wedge I_{O_{2h}}(k_1k_2) \\ F_{(O_{1h} \circ O_{2h})}(k_1l)(k_2l) = (F_{O_{1h}} \circ F_{O_{2h}})(k_1l)(k_2l) = F_{O_2}(l) \vee F_{O_{2h}}(k_1k_2) \end{cases}$
for all $l \in P_2$, $(k_1k_2) \in P_{1h}$,

(iv) $\begin{cases} T_{(O_{1h} \circ O_{2h})}(k_1l_1)(k_2l_2) = (T_{O_{1h}} \circ T_{O_{2h}})(k_1l_1)(k_2l_2) = T_{O_{1h}}(k_1k_2) \wedge T_{O_2}(l_1) \wedge T_{O_2}(l_2) \\ I_{(O_{1h} \circ O_{2h})}(k_1l_1)(k_2l_2) = (I_{O_{1h}} \circ I_{O_{2h}})(k_1l_1)(k_2l_2) = I_{O_{1h}}(k_1k_2) \wedge I_{O_2}(l_1) \wedge I_{O_2}(l_2) \\ F_{(O_{1h} \circ O_{2h})}(k_1l_1)(k_2l_2) = (F_{O_{1h}} \circ F_{O_{2h}})(k_1l_1)(k_2l_2) = F_{O_{1h}}(k_1k_2) \vee F_{O_2}(l_1) \vee F_{O_2}(l_2) \end{cases}$
for all $(k_1k_2) \in P_{1h}$, $(l_1l_2) \in P_{2h}$ such that $l_1 \neq l_2$.

Example 2.18. Composition of SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2.2 is defined as $\check{G}_{i1} \circ \check{G}_{i2} = \{O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}\}$ and is represented in Fig. 2.7.
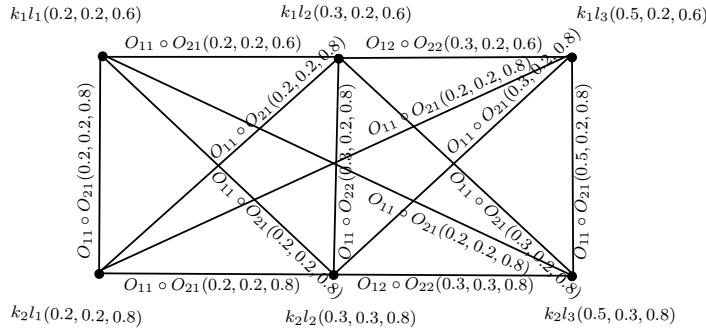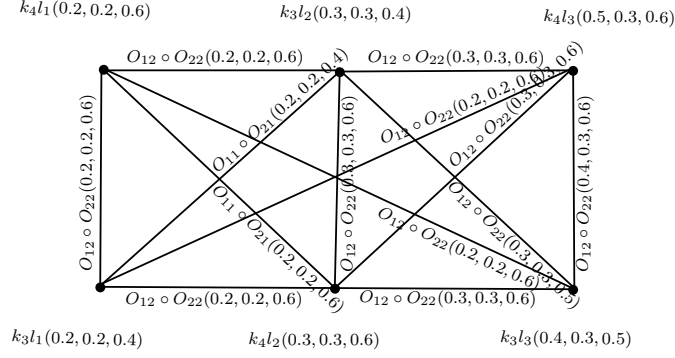
Figure 2.7: $\check{G}_{i1} \circ \check{G}_{i2}$

Theorem 2.19. *The composition* $\check{G}_{i1} \circ \check{G}_{i2} = (O_1 \circ O_2, O_{11} \circ O_{21}, O_{12} \circ O_{22}, \ldots, O_{1r} \circ O_{2r})$ *of two SVINSGSs of GRSs* $\check{G}_1$ *and* $\check{G}_2$ *is SVINSGS of* $\check{G}_1 \circ \check{G}_2$.

Proof. We consider three cases:

Case 1. For $k \in P_1$, $l_1 l_2 \in P_{2h}$

$$
\begin{aligned}
T_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= T_{O_1}(k) \wedge T_{O_{2h}}(l_1 l_2) \\
&\le T_{O_1}(k) \wedge [T_{O_2}(l_1) \wedge T_{O_2}(l_2)] \\
&= [T_{O_1}(k) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k) \wedge T_{O_2}(l_2)] \\
&= T_{(O_1 \circ O_2)}(kl_1) \wedge T_{(O_1 \circ O_2)}(kl_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= I_{O_1}(k) \wedge I_{O_{2h}}(l_1 l_2) \\
&\le I_{O_1}(k) \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)] \\
&= [I_{O_1}(k) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k) \wedge I_{O_2}(l_2)] \\
&= I_{(O_1 \circ O_2)}(kl_1) \wedge I_{(O_1 \circ O_2)}(kl_2),
\end{aligned}
$$

$$
\begin{aligned}
F_{(O_{1h} \circ O_{2h})}((kl_1)(kl_2)) &= F_{O_1}(k) \vee F_{O_{2h}}(l_1 l_2) \\
&\le F_{O_1}(k) \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)] \\
&= [F_{O_1}(k) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k) \vee F_{O_2}(l_2)] \\
&= F_{(O_1 \circ O_2)}(kl_1) \vee F_{(O_1 \circ O_2)}(kl_2),
\end{aligned}
$$

for $kl_1, kl_2 \in P_1 \circ P_2$.

Case 2. For $k \in P_2$, $l_1 l_2 \in P_{1h}$

$$
\begin{aligned}
T_{(O_{1h} \circ O_{2h})}((l_1 k)(l_2 k)) &= T_{O_2}(k) \wedge T_{O_{1h}}(l_1 l_2) \\
&\le T_{O_2}(k) \wedge [T_{O_1}(l_1) \wedge T_{O_1}(l_2)] \\
&= [T_{O_2}(k) \wedge T_{O_1}(l_1)] \wedge [T_{O_2}(k) \wedge T_{O_1}(l_2)] \\
&= T_{(O_1 \circ O_2)}(l_1 k) \wedge T_{(O_1 \circ O_2)}(l_2 k),
\end{aligned}
$$

$$
\begin{aligned}
I_{(O_{1h} \circ O_{2h})}((l_1 k)(l_2 k)) &= I_{O_2}(k) \wedge I_{O_{1h}}(l_1 l_2) \\
&\le I_{O_2}(k) \wedge [I_{O_1}(l_1) \wedge I_{O_1}(l_2)] \\
&= [I_{O_2}(k) \wedge I_{O_1}(l_1)] \wedge [I_{O_2}(k) \wedge I_{O_1}(l_2)] \\
&= I_{(O_1 \circ O_2)}(l_1 k) \wedge I_{(O_1 \circ O_2)}(l_2 k),
\end{aligned}
$$

$$F_{(O_{1h} \circ O_{2h})}((l_1 k)(l_2 k)) = F_{O_2}(k) \vee F_{O_{1h}}(l_1 l_2)$$
$$\leq F_{O_2}(k) \vee [F_{O_1}(l_1) \vee F_{O_1}(l_2)]$$
$$= [F_{O_2}(k) \vee F_{O_1}(l_1)] \vee [F_{O_2}(k) \vee F_{O_1}(l_2)]$$
$$= F_{(O_1 \circ O_2)}(l_1 k) \vee F_{(O_1 \circ O_2)}(l_2 k),$$

for $l_1 k, l_2 k \in P_1 \circ P_2$.

Case 3. For $k_1 k_2 \in P_{1h}, l_1, l_2 \in P_2$ such that $l_1 \neq l_2$

$$T_{(O_{1h} \circ O_{2h})}((k_1 l_1)(k_2 l_2)) = T_{O_{1h}}(k_1 k_2) \wedge T_{O_2}(l_1) \wedge T_{O_2}(l_2)$$
$$\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \wedge T_{O_2}(l_1) \wedge T_{O_2}(l_2)$$
$$= [T_{O_1}(k_1) \wedge T_{O_2}(l_1)] \wedge [T_{O_1}(k_2) \wedge T_{O_2}(l_2)]$$
$$= T_{(O_1 \circ O_2)}(k_1 l_1) \wedge T_{(O_1 \circ O_2)}(k_2 l_2),$$

$$I_{(O_{1h} \circ O_{2h})}((k_1 l_1)(k_2 l_2)) = I_{O_{1h}}(k_1 k_2) \wedge I_{O_2}(l_1) \wedge I_{O_2}(l_2)$$
$$\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \wedge [I_{O_2}(l_1) \wedge I_{O_2}(l_2)]$$
$$= [I_{O_1}(k_1) \wedge I_{O_2}(l_1)] \wedge [I_{O_1}(k_2) \wedge I_{O_2}(l_2)]$$
$$= I_{(O_1 \circ O_2)}(k_1 l_1) \wedge I_{(O_1 \circ O_2)}(k_2 l_2),$$

$$F_{(O_{1h} \circ O_{2h})}((k_1 l_1)(k_2 l_2)) = F_{O_{1h}}(k_1 k_2) \vee F_{O_2}(l_1) \vee F_{O_2}(l_2)$$
$$\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \vee [F_{O_2}(l_1) \vee F_{O_2}(l_2)]$$
$$= [F_{O_1}(k_1) \vee F_{O_2}(l_1)] \vee [F_{O_1}(k_2) \vee F_{O_2}(l_2)]$$
$$= F_{(O_1 \circ O_2)}(k_1 l_1) \vee F_{(O_1 \circ O_2)}(k_2 l_2),$$

for $k_1 l_1, k_2 l_2 \in P_1 \circ P_2$.
All cases holds for $h = 1, 2, \ldots, r$. This gives proof. $\qquad \square$

Definition 2.20. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be SVINSGSs. *Union* of $\check{G}_{i1}$ and $\check{G}_{i2}$, denoted by

$$\check{G}_{i1} \cup \check{G}_{i2} = (O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}, \ldots, O_{1r} \cup O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1 \cup O_2)}(k) = (T_{O_1} \cup T_{O_2})(k) = T_{O_1}(k) \vee T_{O_2}(k) \\ I_{(O_1 \cup O_2)}(k) = (I_{O_1} \cup I_{O_2})(k) = I_{O_1}(k) \vee I_{O_2}(k) \\ F_{(O_1 \cup O_2)}(k) = (F_{O_1} \cup F_{O_2})(k) = F_{O_1}(k) \wedge F_{O_2}(k) \end{cases}$
for all $k \in P_1 \cup P_2$,

(ii) $\begin{cases} T_{(O_{1h} \cup O_{2h})}(kl) = (T_{O_{1h}} \cup T_{O_{2h}})(kl) = T_{O_{1h}}(kl) \vee T_{O_{2h}}(kl) \\ I_{(O_{1h} \cup O_{2h})}(kl) = (I_{O_{1h}} \cup I_{O_{2h}})(kl) = I_{O_{1h}}(kl) \vee I_{O_{2h}}(kl) \\ F_{(O_{1h} \cup O_{2h})}(kl) = (F_{O_{1h}} \cup F_{O_{2h}})(kl) = F_{O_{1h}}(kl) \wedge F_{O_{2h}}(kl) \end{cases}$
for all $(kl) \in P_{1h} \cup P_{2h}$.

Example 2.21. Union of two SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2.2 is defined as $\check{G}_{i1} \cup \check{G}_{i2} = \{O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}\}$ and is represented in Fig. 2.8.
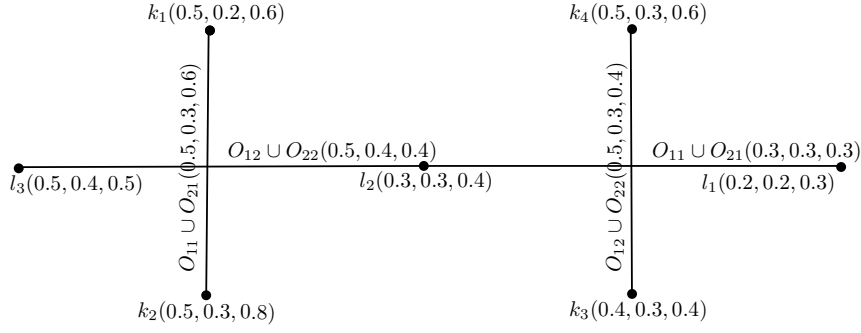
Figure 2.8: $\check{G}_{i1} \cup \check{G}_{i2}$

Theorem 2.22. *Union* $\check{G}_{i1} \cup \check{G}_{i2} = (O_1 \cup O_2, O_{11} \cup O_{21}, O_{12} \cup O_{22}, \ldots, O_{1r} \cup O_{2r})$ *of two SVINSGSs of the GRSs* $\check{G}_1$ *and* $\check{G}_2$ *is SVINSGS of* $\check{G}_1 \cup \check{G}_2$.

Proof. Let $k_1 k_2 \in P_{1h} \cup P_{2h}$. There are two cases:

Case 1. For $k_1, k_2 \in P_1$, by definition 2.20, $T_{O_2}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1 k_2) = 0$, $I_{O_2}(k_1) = I_{O_2}(k_2) = I_{O_{2h}}(k_1 k_2) = 0$, $F_{O_2}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1 k_2) = 0$, so

$$
\begin{aligned}
T_{(O_{1h} \cup O_{2h})}(k_1 k_2) &= T_{O_{1h}}(k_1 k_2) \vee T_{O_{2h}}(k_1 k_2) \\
&= T_{O_{1h}}(k_1 k_2) \vee 0 \\
&\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \vee 0 \\
&= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_1}(k_2) \vee 0] \\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)] \\
&= T_{(O_1 \cup O_2)}(k_1) \wedge T_{(O_1 \cup O_2)}(k_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(O_{1h} \cup O_{2h})}(k_1 k_2) &= I_{O_{1h}}(k_1 k_2) \vee I_{Q_{2h}}(k_1 k_2) \\
&= I_{O_{1h}}(k_1 k_2) \vee 0 \\
&\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \vee 0 \\
&= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_1}(k_2) \vee 0] \\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)] \\
&= I_{(O_1 \cup O_2)}(k_1) \wedge I_{(O_1 \cup O_2)}(k_2),
\end{aligned}
$$

$$
\begin{aligned}
F_{(O_{1h} \cup O_{2h})}(k_1 k_2) &= F_{O_{1h}}(k_1 k_2) \wedge F_{O_{2h}}(k_1 k_2) \\
&= F_{O_{1i}}(k_1 k_2) \wedge 0 \\
&\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \wedge 0 \\
&= [F_{O_1}(k_1) \wedge 0] \vee [F_{O_1}(k_2) \wedge 0] \\
&= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)] \\
&= F_{(O_1 \cup O_2)}(k_1) \vee F_{(O_1 \cup O_2)}(k_2),
\end{aligned}
$$

for $k_1, k_2 \in P_1 \cup P_2$.

Case 2. For $k_1, k_2 \in P_2$, by definition 2.20, $T_{O_1}(k_1) = T_{O_1}(k_2) = T_{O_{1h}}(k_1 k_2) = 0$, $I_{O_1}(k_1) = I_{O_1}(k_2) = I_{O_{1h}}(k_1 k_2) = 0$, $F_{O_1}(k_1) = F_{O_1}(q_2) = F_{O_{1h}}(k_1 k_2) = 0$, so

$$
\begin{aligned}
T_{(O_{1h} \cup O_{2h})}(k_1 k_2) &= T_{O_{1h}}(k_1 k_2) \vee T_{O_{2i}}(k_1 k_2) \\
&= T_{O_{2i}}(k_1 k_2) \vee 0 \\
&\leq [T_{O_2}(k_1) \wedge T_{O_2}(k_2)] \vee 0 \\
&= [T_{O_2}(k_1) \vee 0] \wedge [T_{O_2}(k_2) \vee 0] \\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)] \\
&= T_{(O_1 \cup O_2)}(k_1) \wedge T_{(O_1 \cup O_2)}(k_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(O_{1h} \cup O_{2h})}(q_1 k_2) &= I_{O_{1h}}(k_1 k_2) \vee I_{O_{2h}}(k_1 k_2) \\
&= I_{O_{2h}}(k_1 k_2) \vee 0 \\
&\leq [I_{O_2}(k_1) \wedge I_{O_2}(k_2)] \vee 0 \\
&= [I_{O_2}(k_1) \vee 0] \wedge [I_{O_2}(k_2) \vee 0] \\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)] \\
&= I_{(O_1 \cup O_2)}(k_1) \wedge I_{(O_1 \cup O_2)}(k_2),
\end{aligned}
$$

$$
\begin{aligned}
F_{(O_{1h} \cup O_{2h})}(k_1 k_2) &= F_{O_{1h}}(k_1 k_2) \wedge F_{O_{2h}}(k_1 k_2) \\
&= F_{O_{2h}}(k_1 k_2) \wedge 0 \\
&\leq [F_{O_2}(k_1) \vee F_{O_2}(k_2)] \wedge 0 \\
&= [F_{O_2}(k_1) \wedge 0] \vee [F_{O_2}(k_2) \wedge 0] \\
&= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)] \\
&= F_{(O_1 \cup O_2)}(k_1) \vee F_{(O_1 \cup O_2)}(k_2),
\end{aligned}
$$

for $k_1, k_2 \in P_1 \cup P_2$.

Both cases hold $\forall h \in \{1, 2, \ldots, r\}$. So, we get proof. $\square$

Theorem 2.23. *Let $\check{G} = (P_1 \cup P_2, P_{11} \cup P_{21}, P_{12} \cup P_{22}, \ldots, P_{1r} \cup P_{2r})$ be the union of two GRSs $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, \ldots, P_{2r})$. Then every SVINSGS $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ of $\check{G}$ is union of the two SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ of GRSs $\check{G}_1$ and $\check{G}_2$, respectively.*

Proof. Firstly, we define O_1, O_2, O_{1h} and O_{2h} for $h \in \{1, 2, \ldots, r\}$ as:
$T_{O_1}(k) = T_O(k), I_{O_1}(k) = I_O(k), F_{O_1}(k) = F_O(k)$, if $k \in P_1$
$T_{O_2}(k) = T_O(k), I_{O_2}(k) = I_O(k), F_{O_2}(k) = F_O(k)$, if $k \in P_2$

$T_{O_{1h}}(k_1 k_2) = T_{O_h}(k_1 k_2), I_{O_{1h}}(k_1 k_2) = I_{O_h}(k_1 k_2), F_{O_{1h}}(k_1 k_2) = F_{O_h}(k_1 k_2)$, if $k_1 k_2 \in P_{1h}$, $T_{O_{2h}}(k_1 k_2) = T_{O_h}(k_1 k_2), I_{O_{2h}}(k_1 k_2) = I_{O_h}(k_1 k_2), F_{O_{2h}}(k_1 k_2) = F_{O_h}(k_1 k_2)$, if $k_1 k_2 \in P_{2h}$. Then $O = O_1 \cup O_2$ and $O_h = O_{1h} \cup O_{2h}$, $h \in \{1, 2, \ldots, r\}$.

Now for $k_1 k_2 \in P_{lh}$, $l = 1, 2$, $h = 1, 2, \ldots, r$
$T_{O_{lh}}(k_1 k_2) = T_{O_h}(k_1 k_2) \leq T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2)$,
$I_{O_{lh}}(k_1 k_2) = I_{O_h}(k_1 k_2) \leq I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2)$,
$F_{O_{lh}}(k_1 k_2) = F_{O_h}(k_1 k_2) \leq F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2)$,
i.e.,
$\check{G}_{il} = (O_l, O_{l1}, O_{l2}, \ldots, O_{lr})$ is a SVINSGS of $\check{G}_l$, $l = 1, 2$.
Thus $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$, a SVINSGS of $\check{G} = \check{G}_1 \cup \check{G}_2$, is union of the two SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$. $\square$

Definition 2.24. Let $\check{G}_{i1} = (O_1, O_{11}, O_{12}, ..., O_{1r})$ and $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ be SVINSGSs and let $P_1 \cap P_2 = \emptyset$. *Join of $\check{G}_{i1}$ and $\check{G}_{i2}$*, denoted by

$$\check{G}_{i1} + \check{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r}),$$

is defined as:

(i) $\begin{cases} T_{(O_1+O_2)}(k) = T_{(O_1 \cup O_2)}(k) \\ I_{(O_1+O_2)}(k) = I_{(O_1 \cup O_2)}(k) \\ F_{(O_1+O_2)}(k) = F_{(O_1 \cup O_2)}(k) \end{cases}$
for all $k \in P_1 \cup P_2$,

(ii) $\begin{cases} T_{(O_{1h}+O_{2h})}(kl) = T_{(O_{1h} \cup O_{2h})}(kl) \\ I_{(O_{1h}+O_{2h})}(kl) = I_{(O_{1h} \cup O_{2h})}(kl) \\ F_{(O_{1h}+O_{2h})}(kl) = F_{(O_{1h} \cup O_{2h})}(kl) \end{cases}$
for all $(kl) \in P_{1h} \cup P_{2h}$,

(iii) $\begin{cases} T_{(O_{1h}+O_{2h})}(kl) = (T_{O_{1h}} + T_{O_{2h}})(kl) = T_{O_1}(k) \wedge T_{O_2}(l) \\ I_{(O_{1h}+O_{2h})}(kl) = (I_{O_{1h}} + I_{O_{2h}})(kl) = I_{O_1}(k) \wedge I_{O_2}(l) \\ F_{(O_{1h}+O_{2h})}(kl) = (F_{O_{1h}} + F_{O_{2h}})(kl) = F_{O_1}(k) \vee F_{O_2}(l) \end{cases}$
for all $k \in P_1$, $l \in P_2$.

Example 2.25. Join of two SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ shown in Fig. 2.2 is defined as $\check{G}_{i1} + \check{G}_{i2} = \{O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}\}$ and is represented in the Fig. 2.9.
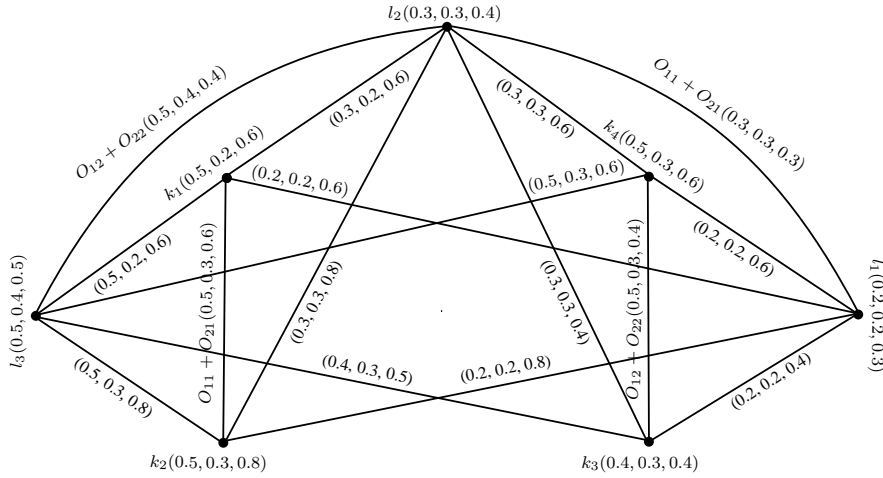
Figure 2.9: $\check{G}_{i1} + \check{G}_{i2}$

Theorem 2.26. *The join $\check{G}_{i1} + \check{G}_{i2} = (O_1 + O_2, O_{11} + O_{21}, O_{12} + O_{22}, \ldots, O_{1r} + O_{2r})$ of two SVINSGSs of GRSs $\check{G}_1$ and $\check{G}_2$ is SVINSGS of $\check{G}_1 + \check{G}_2$.*

Proof. Let $k_1 k_2 \in P_{1h} + P_{2h}$. There are three cases:

Case 1. For $k_1, k_2 \in P_1$, by definition 2.24, $T_{O_2}(k_1) = T_{O_2}(k_2) = T_{O_{2h}}(k_1 k_2) = 0$, $I_{O_2}(k_1) = I_{O_2}(k_2) = $

$I_{O_{2h}}(k_1 k_2) = 0$, $F_{O_2}(k_1) = F_{O_2}(k_2) = F_{O_{2h}}(k_1 k_2) = 0$, so,

$$
\begin{aligned}
T_{(O_{1h}+O_{2h})}(k_1 k_2) &= T_{O_{1h}}(k_1 k_2) \vee T_{O_{2h}}(k_1 k_2) \\
&= T_{O_{1h}}(k_1 k_2) \vee 0 \\
&\leq [T_{O_1}(k_1) \wedge T_{O_1}(k_2)] \vee 0 \\
&= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_1}(q_2) \vee 0] \\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)] \\
&= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(O_{1h}+O_{2h})}(k_1 k_2) &= I_{O_{1h}}(k_1 k_2) \vee I_{O_{2h}}(k_1 k_2) \\
&= I_{O_{1h}}(k_1 k_2) \vee 0 \\
&\leq [I_{O_1}(k_1) \wedge I_{O_1}(k_2)] \vee 0 \\
&= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_1}(k_2) \vee 0] \\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)] \\
&= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),
\end{aligned}
$$

$$
\begin{aligned}
F_{(O_{1h}+O_{2h})}(k_1 k_2) &= F_{O_{1h}}(k_1 k_2) \wedge F_{O_{2h}}(k_1 k_2) \\
&= F_{O_{1h}}(k_1 k_2) \wedge 0 \\
&\leq [F_{O_1}(k_1) \vee F_{O_1}(k_2)] \wedge 0 \\
&= [F_{O_1}(k_1) \wedge 0] \vee [F_{O_1}(k_2) \wedge 0] \\
&= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)] \\
&= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2),
\end{aligned}
$$

for $k_1, k_2 \in P_1 + P_2$.

Case 2. For $k_1, k_2 \in P_2$, by definition 2.24, $T_{O_1}(k_1) = T_{O_1}(k_2) = T_{O_{1h}}(k_1 k_2) = 0$, $I_{O_1}(k_1) = I_{O_1}(k_2) = I_{O_{1h}}(k_1 k_2) = 0$, $F_{O_1}(k_1) = F_{O_1}(k_2) = F_{O_{1h}}(k_1 k_2) = 0$, so

$$
\begin{aligned}
T_{(O_{1h}+O_{2h})}(k_1 k_2) &= T_{O_{1i}}(k_1 k_2) \vee T_{O_{2i}}(k_1 k_2) \\
&= T_{O_{2i}}(k_1 k_2) \vee 0 \\
&\leq [T_{O_2}(k_1) \wedge T_{O_2}(k_2)] \vee 0 \\
&= [T_{O_2}(k_1) \vee 0] \wedge [T_{O_2}(k_2) \vee 0] \\
&= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_1}(k_2) \vee T_{O_2}(k_2)] \\
&= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),
\end{aligned}
$$

$$
\begin{aligned}
I_{(O_{1h}+O_{2h})}(k_1 k_2) &= I_{O_{1h}}(k_1 k_2) \vee I_{O_{2h}}(k_1 k_2) \\
&= I_{O_{2h}}(k_1 k_2) \vee 0 \\
&\leq [I_{O_2}(k_1) \wedge I_{O_2}(k_2)] \vee 0 \\
&= [I_{O_2}(k_1) \vee 0] \wedge [I_{O_2}(k_2) \vee 0] \\
&= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_1}(k_2) \vee I_{O_2}(k_2)] \\
&= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),
\end{aligned}
$$

$$F_{(O_{1h}+O_{2h})}(k_1k_2) = F_{O_{1h}}(k_1k_2) \wedge F_{O_{2h}}(k_1k_2)$$
$$= F_{O_{2h}}(k_1k_2) \wedge 0$$
$$\leq [F_{O_2}(k_1) \vee F_{O_2}(k_2)] \wedge 0$$
$$= [F_{O_2}(k_1) \wedge 0] \vee [F_{O_2}(k_2) \wedge 0]$$
$$= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_1}(k_2) \wedge F_{O_2}(k_2)]$$
$$= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2),$$

for $q_1, q_2 \in P_1 + P_2$.

Case 3. For $k_1 \in P_1$, $k_2 \in P_2$, by definition 2.24,
$T_{O_1}(k_2) = T_{O_2}(k_1) = 0$, $I_{O_1}(k_2) = I_{O_2}(k_1) = 0$, $F_{O_1}(k_2) = F_{O_2}(k_1) = 0$, so

$$T_{(O_{1h}+O_{2h})}(k_1k_2) = T_{O_1}(q_1) \wedge T_{O_2}(k_2)$$
$$= [T_{O_1}(k_1) \vee 0] \wedge [T_{O_2}(k_2) \vee 0]$$
$$= [T_{O_1}(k_1) \vee T_{O_2}(k_1)] \wedge [T_{O_2}(k_2) \vee T_{O_1}(k_2)]$$
$$= T_{(O_1+O_2)}(k_1) \wedge T_{(O_1+O_2)}(k_2),$$

$$I_{(O_{1h}+O_{2h})}(k_1k_2) = I_{O_1}(k_1) \wedge I_{O_2}(k_2)$$
$$= [I_{O_1}(k_1) \vee 0] \wedge [I_{O_2}(k_2) \vee 0]$$
$$= [I_{O_1}(k_1) \vee I_{O_2}(k_1)] \wedge [I_{O_2}(k_2) \vee I_{O_1}(k_2)]$$
$$= I_{(O_1+O_2)}(k_1) \wedge I_{(O_1+O_2)}(k_2),$$

$$F_{(O_{1h}+O_{2h})}(k_1k_2) = F_{O_1}(k_1) \vee F_{O_2}(k_2)$$
$$= [F_{O_1}(k_1) \wedge 0] \vee [F_{O_2}(k_2) \wedge 0]$$
$$= [F_{O_1}(k_1) \wedge F_{O_2}(k_1)] \vee [F_{O_2}(k_2) \wedge F_{O_1}(k_2)]$$
$$= F_{(O_1+O_2)}(k_1) \vee F_{(O_1+O_2)}(k_2),$$

for $k_1, k_2 \in P_1 + P_2$.

All these cases hold $\forall h \in \{1, 2, \ldots, r\}$. We get proof. $\square$

Theorem 2.27. *If $\check{G} = (P_1 + P_2, P_{11} + P_{21}, P_{12} + P_{22}, \ldots, P_{1r} + P_{2r})$ is join of the two GRSs $\check{G}_1 = (P_1, P_{11}, P_{12}, ..., P_{1r})$ and $\check{G}_2 = (P_2, P_{21}, P_{22}, ..., P_{2r})$. Then each strong SVINSGS $\check{G}_i = (O, O_1, O_2, ..., O_r)$ of $\check{G}$, is join of the two strong SVINSGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ of GRSs $\check{G}_1$ and $\check{G}_2$, respectively.*

Proof. We define O_l and O_{lh} for $l = 1, 2$ and $h = 1, 2, \ldots, r$ as:
$T_{O_l}(k) = T_O(k), I_{O_k}(k) = I_O(k), F_{O_l}(k) = F_O(k)$, if $k \in P_l$
$T_{O_{lh}}(k_1k_2) = T_{O_h}(k_1k_2), I_{O_{lh}}(k_1k_2) = I_{O_h}(k_1k_2), F_{O_{lh}}(k_1k_2) = F_{O_h}(k_1k_2)$, if $k_1k_2 \in P_{lh}$.

Now for $k_1k_2 \in P_{lh}$, $l = 1, 2$, $h = 1, 2, \ldots, r$
$T_{O_{lh}}(k_1k_2) = T_{O_h}(k_1k_2) = T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2)$,
$I_{O_{lh}}(k_1k_2) = I_{O_h}(k_1k_2) = I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2)$,
$F_{O_{lh}}(k_1k_2) = F_{O_h}(k_1k_2) = F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2)$,
i.e.,
$\check{G}_{il} = (O_l, O_{l1}, O_{l2}, ..., O_{lr})$ is strong SVINSGS of $\check{G}_l$, $l = 1, 2$.
Moreover, $\check{G}_i$ is join of $\check{G}_{i1}$ and $\check{G}_{i2}$ as shown:
According to the definitions 2.20 and 2.24, $O = O_1 \cup O_2 = O_1 + O_2$ and $O_h = O_{1h} \cup O_{2h} = O_{1h} + O_{2h}$,

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$\forall k_1 k_2 \in P_{1h} \cup P_{2h}$.

When $k_1 k_2 \in P_{1h} + P_{2h} \ (P_{1h} \cup P_{2h})$, i.e., $k_1 \in P_1$ and $k_2 \in P_2$

$T_{O_h}(k_1 k_2) = T_O(k_1) \wedge T_O(k_2) = T_{O_l}(k_1) \wedge T_{O_l}(k_2) = T_{(O_{1h}+O_{2h})}(k_1 k_2)$,

$I_{O_h}(k_1 k_2) = I_O(k_1) \wedge I_O(k_2) = I_{O_l}(k_1) \wedge I_{O_l}(k_2) = I_{(O_{1h}+O_{2h})}(k_1 k_2)$,

$F_{O_h}(k_1 k_2) = F_O(k_1) \vee F_O(k_2) = F_{O_l}(k_1) \vee F_{O_l}(k_2) = F_{(O_{1h}+O_{2h})}(k_1 k_2)$,

when $k_1 \in P_2$, $k_2 \in P_1$, we get similar calculations. It's true for $h = 1, 2, \ldots, r$. So, we get proof. $\qquad\square$

3   Decision-Making of Eminent Trade Among Poor Countries

According to IMF data, 1.75 billion people are living in poverty, their living is estimated to be less than two dollars a day. Poverty changes by region, for example in Europe it is 3%, and in the Sub-Saharan Africa it is up to 65%. We rank the countries of the World as poor or rich, using their GDP per capita as scale. Poor countries are trying to catch up with rich or developed countries. But this ratio is very small, that's why trade of poor countries among themselves is very important. There are different types of trade among poor countries, for example: agricultural or food items, raw minerals, medicines, textile materials, industrials goods etc. Using SVINSGS, we can estimate between any two poor countries which trade is comparatively stronger than others. Moreover, we can decide(judge) which country has large number of resources for particular type of goods and better circumstances for its trade. We can figure out, for which trade, an external investor can invest his money in these poor countries. Further, it will be easy to judge that in which field these poor countries are trying to be better, and can be helped. It will also help in deciding that in which trade they are weak, and should be facilitated, so that they can be independent and improve their living standards.

We consider a set of nine poor countries in the World:

$P = \{Congo, Liberia, Burundi, Tanzania, Ugenda, SierriaLeone, Zimbabwe, Kenya, Zambia\}$. Let O be the SVINS set on P, as defined in Table 1.

Table 1: SVINS set O of nine poor countries on globe

Poor Country	T	I	F
Congo	0.5	0.3	0.2
Liberia	0.4	0.4	0.3
Burundi	0.4	0.4	0.4
Tanzania	0.5	0.5	0.4
Uganda	0.4	0.4	0.5
Sierra Leone	0.5	0.4	0.4
Zimbabwe	0.3	0.4	0.4
Kenya	0.5	0.3	0.3
Zambia	0.4	0.4	0.4

In Table 1, symbol T demonstrates the positive aspects of that poor country, symbol I indicates its negative aspects, whereas F denotes the percentage of ambiguity of its problems for the World. Let we use following alphabets for country names:

CO = Congo, L = Liberia, B = Burundi, T = Tanzania, U = Uganda, SL = Sierra Leone, ZI = Zimbabwe, K = Kenya, ZA = Zambia.

For every pair of poor countries in set P, different trades with their T, I and F values are demonstrated in Tables $2 - 8$.

Table 2: SVINS set of different types of trade between Congo and other poor countries in P

Type of trade	(CO, L)	(CO, B)	(CO, T)	(CO, U)	(CO, K)
Food items	(0.1, 0.2, 0.3)	(0.4, 0.2, 0.1)	(0.2, 0.1, 0.4)	(0.4, 0.3, 0.5)	(0.2, 0.1, 0.3)
Chemicals	(0.2, 0.4, 0.3)	(0.1, 0.2, 0.1)	(0.1, 0.2, 0.4)	(0.3, 0.2, 0.4)	(0.5, 0.1, 0.1)
Oil	(0.4, 0.2, 0.1)	(0.4, 0.3, 0.2)	(0.5, 0.1, 0.2)	(0.4, 0.2, 0.2)	(0.5, 0.3, 0.1)
Raw minerals	(0.3, 0.1, 0.1)	(0.4, 0.3, 0.3)	(0.4, 0.2, 0.2)	(0.4, 0.1, 0.2)	(0.5, 0.1, 0.1)
Textile products	(0.2, 0.3, 0.3)	(0.1, 0.3, 0.4)	(0.1, 0.2, 0.4)	(0.1, 0.3, 0.2)	(0.2, 0.1, 0.3)
Gold and diamonds	(0.4, 0.1, 0.1)	(0.4, 0.2, 0.2)	(0.2, 0.2, 0.4)	(0.2, 0.2, 0.4)	(0.1, 0.3, 0.3)

Table 3: SVINS set of different types of trade between Liberia and other poor countries in P

Type of trade	(L, B)	(L, T)	(L, U)	(L, SL)	(L, ZI)
Food items	(0.4, 0.2, 0.2)	(0.4, 0.3, 0.2)	(0.3, 0.3, 0.4)	(0.3, 0.3, 0.3)	(0.2, 0.3, 0.3)
Chemicals	(0.2, 0.2, 0.4)	(0.1, 0.4, 0.3)	(0.3, 0.3, 0.3)	(0.2, 0.2, 0.4)	(0.1, 0.3, 0.3)
Oil	(0.1, 0.1, 0.4)	(0.2, 0.3, 0.3)	(0.1, 0.1, 0.4)	(0.2, 0.4, 0.3)	(0.2, 0.2, 0.3)
Raw minerals	(0.3, 0.1, 0.3)	(0.2, 0.2, 0.3)	(0.2, 0.1, 0.4)	(0.3, 0.2, 0.3)	(0.2, 0.1, 0.3)
Textile products	(0.1, 0.3, 0.4)	(0.1, 0.3, 0.3)	(0.2, 0.1, 0.3)	(0.1, 0.2, 0.3)	(0.2, 0.2, 0.3)
Gold and diamonds	(0.2, 0.1, 0.4)	(0.2, 0.1, 0.3)	(0.3, 0.1, 0.3)	(0.4, 0.1, 0.1)	(0.3, 0.1, 0.1)

Table 4: SVINS set of different types of trade between Burundi and other poor countries in P

Type of trade	(B, T)	(B, U)	(B, SL)	(B, ZI)	(B, K)
Food items	(0.3, 0.2, 0.2)	(0.4, 0.1, 0.2)	(0.3, 0.3, 0.1)	(0.3, 0.3, 0.2)	(0.3, 0.2, 0.2)
Chemicals	(0.1, 0.2, 0.3)	(0.2, 0.1, 0.3)	(0.2, 0.4, 0.3)	(0.3, 0.4, 0.3)	(0.3, 0.3, 0.1)
Oil	(0.1, 0.1, 0.4)	(0.2, 0.3, 0.4)	(0.2, 0.4, 0.3)	(0.2, 0.2, 0.5)	(0.1, 0.3, 0.4)
Raw minerals	(0.2, 0.1, 0.3)	(0.4, 0.2, 0.3)	(0.4, 0.2, 0.4)	(0.3, 0.2, 0.2)	(0.4, 0.2, 0.2)
Textile products	(0.3, 0.1, 0.1)	(0.2, 0.4, 0.3)	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.1)	(0.4, 0.1, 0.2)
Gold and diamonds	(0.3, 0.2, 0.3)	(0.3, 0.4, 0.3)	(0.1, 0.4, 0.2)	(0.2, 0.4, 0.2)	(0.2, 0.3, 0.4)

Table 5: SVINS set of different types of trade between Tanzania and other poor countries in P

Type of trade	(T, U)	(T, SL)	(T, ZI)	(T, K)	(T,ZA)
Food items	(0.4, 0.2, 0.1)	(0.5, 0.1, 0.1)	(0.3, 0.1, 0.2)	(0.4, 0.3, 0.2)	(0.3, 0.2, 0.2)
Chemicals	(0.2, 0.3, 0.3)	(0.2, 0.3, 0.4)	(0.2, 0.3, 0.3)	(0.4, 0.1, 0.4)	(0.3, 0.4, 0.4)
Oil	(0.1, 0.3, 0.3)	(0.4, 0.1, 0.3)	(0.3, 0.4, 0.2)	(0.2, 0.3, 0.3)	(0.1, 0.3, 0.3)
Raw minerals	(0.3, 0.3, 0.4)	(0.4, 0.3, 0.3)	(0.3, 0.2, 0.1)	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.3)
Textile products	(0.2, 0.4, 0.3)	(0.2, 0.4, 0.4)	(0.1, 0.3, 0.4)	(0.2, 0.3, 0.2)	(0.4, 0.1, 0.2)
Gold and diamonds	(0.3, 0.4, 0.3)	(0.4, 0.3, 0.4)	(0.3, 0.1, 0.1)	(0.2, 0.2, 0.2)	(0.4, 0.3, 0.3)

Table 6: SVINS set of different types of trade between Sierra Leone and other poor countries in P

Type of trade	(SL, ZI)	(SL, K)	(SL, ZA)	(SL, CO)	(L, K)
Food items	(0.3, 0.3, 0.2)	(0.4, 0.2, 0.1)	(0.2, 0.4, 0.3)	(0.5, 0.1, 0.1)	(0.4, 0.1, 0.2)
Chemicals	(0.2, 0.3, 0.4)	(0.3, 0.2, 0.2)	(0.2, 0.4, 0.4)	(0.2, 0.2, 0.3)	(0.2, 0.3, 0.3)
Oil	(0.1, 0.3, 0.4)	(0.2, 0.2, 0.3)	(0.3, 0.4, 0.2)	(0.5, 0.2, 0.1)	(0.3, 0.3, 0.3)
Raw minerals	(0.3, 0.2, 0.2)	(0.5, 0.2, 0.1)	(0.3, 0.1, 0.1)	(0.3, 0.3, 0.3)	(0.4, 0.1, 0.2)
Textile products	(0.2, 0.4, 0.2)	(0.3, 0.2, 0.3)	(0.2, 0.2, 0.4)	(0.2, 0.2, 0.3)	(0.3, 0.3, 0.2)
Gold and diamonds	(0.3, 0.1, 0.1)	(0.1, 0.2, 0.4)	(0.2, 0.3, 0.3)	(0.4, 0.1, 0.2)	(0.3, 0.2, 0.3)

Table 7: SVINS set of different types of trade between Zimbabwe and other poor countries in P

Type of trade	(ZI, K)	(ZI, ZA)	(ZI, U)	(ZI, CO)
Food items	(0.3, 0.2, 0.2)	(0.3, 0.1, 0.1)	(0.3, 0.1, 0.1)	(0.2, 0.1, 0.1)
Chemicals	(0.3, 0.3, 0.2)	(0.2, 0.4, 0.3)	(0.3, 0.2, 0.2)	(0.2, 0.1, 0.2)
Oil	(0.1, 0.3, 0.3)	(0.1, 0.4, 0.4)	(0.3, 0.2, 0.1)	(0.3, 0.1, 0.1)
Raw minerals	(0.3, 0.1, 0.2)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.3)	(0.2, 0.3, 0.1)
Textile products	(0.2, 0.2, 0.2)	(0.2, 0.4, 0.3)	(0.2, 0.3, 0.3)	(0.2, 0.3, 0.1)
Gold and diamonds	(0.3, 0.3, 0.1)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.1)

Table 8: SVINS set of different types of trade between Zambia and other poor countries in P

Type of trade	(ZA, CO)	(ZA, L)	(ZA, B)	(ZA, K)
Food items	(0.3, 0.1, 0.2)	(0.3, 0.1, 0.2)	(0.4, 0.2, 0.1)	(0.3, 0.1, 0.3)
Chemicals	(0.2, 0.2, 0.2)	(0.2, 0.2, 0.1)	(0.3, 0.2, 0.2)	(0.3, 0.1, 0.1)
Oil	(0.4, 0.1, 0.1)	(0.2, 0.1, 0.1)	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.2)
Raw minerals	(0.3, 0.1, 0.1)	(0.4, 0.1, 0.1)	(0.4, 0.2, 0.2)	(0.4, 0.1, 0.1)
Textile products	(0.2, 0.2, 0.2)	(0.2, 0.2, 0.3)	(0.2, 0.3, 0.2)	(0.3, 0.1, 0.2)
Gold and diamonds	(0.1, 0.2, 0.4)	(0.4, 0.3, 0.2)	(0.2, 0.3, 0.2)	(0.3, 0.2, 0.1)

Many relations can be defined on the set P, we define following relations on set P as:

P_1 = Food items, P_2 = Chemicals, P_3 = Oil, P_4 = Raw minerals, P_5 = Textile products, P_6 = Gold and diamonds, such that $(P, P_1, P_2, P_3, P_4, P_5, P_6)$ is a GRS. Any element of a relation demonstrates a particular trade between those two poor countries. As $(P, P_1, P_2, P_3, P_4, P_5, P_6)$ is GRS, that's why any element can appear in only one relation. Therefore, any element will be considered in that relation, whose value of T is high, and values of I, F are comparatively low, using data of above tables.

Write down T, I and F values of the elements in relations according to above data, such that O_1, O_2, O_3, O_4, O_5, O_6 are SVINS sets on relations P_1, P_2, P_3, P_4, P_5, P_6, respectively.

Let $P_1 = \{(Burundi, Congo), (SierraLeone, Congo), (Burundi, Zambia)\}$, $P_2 = \{(Kenya, Congo)\}$, $P_3 = \{(Congo, Zambia), (Congo, Tanzania), (Zimbabwe, Congo)\}$, $P_4 = \{(Congo, Uganda), (SierraLeone, Kenya), (Zambia, Kenya)\}$, $P_5 = \{(Burundi, Zimbabwe), (Tanzania, Burundi)\}$, $P_6 = \{(SierraLeone, Liberia), (Uganda, SierraLeone), (Zimbabwe, SierraLeone)\}$.

Let $O_1 = \{((B, CO), 0.4, 0.2, 0.1), ((SL, CO), 0.5, 0.1, 0.1), ((B, ZA), 0.4, 0.2, 0.1)\}$, $O_2 = \{((K, CO), 0.5, 0.1, 0.1)\}$, $O_3 = \{((CO, ZA), 0.4, 0.1, 0.1), ((CO, T), 0.5, 0.1, 0.2), ((ZI, CO), 0.3, 0.1, 0.1)\}$, $O_4 = \{((CO, U), 0.4, 0.1, 0.2), ((SL, K), 0.5, 0.2, 0.1), ((ZA, K), 0.4, 0.1, 0.1)\}$, $O_5 = \{((B, ZI), 0.3, 0.2, 0.1), ((T, B), 0.3, 0.1, 0.1)\}$, $O_6 = \{((SL, L), 0.4, 0.1, 0.1), ((U, SL), 0.4, 0.2, 0.1), ((ZI, SL), 0.3, 0.1, 0.1)\}$.

Obviously, $(O, O_1, O_2, O_3, O_4, O_5, O_6)$ is a SVINSGS as shown in Fig. 3.1.
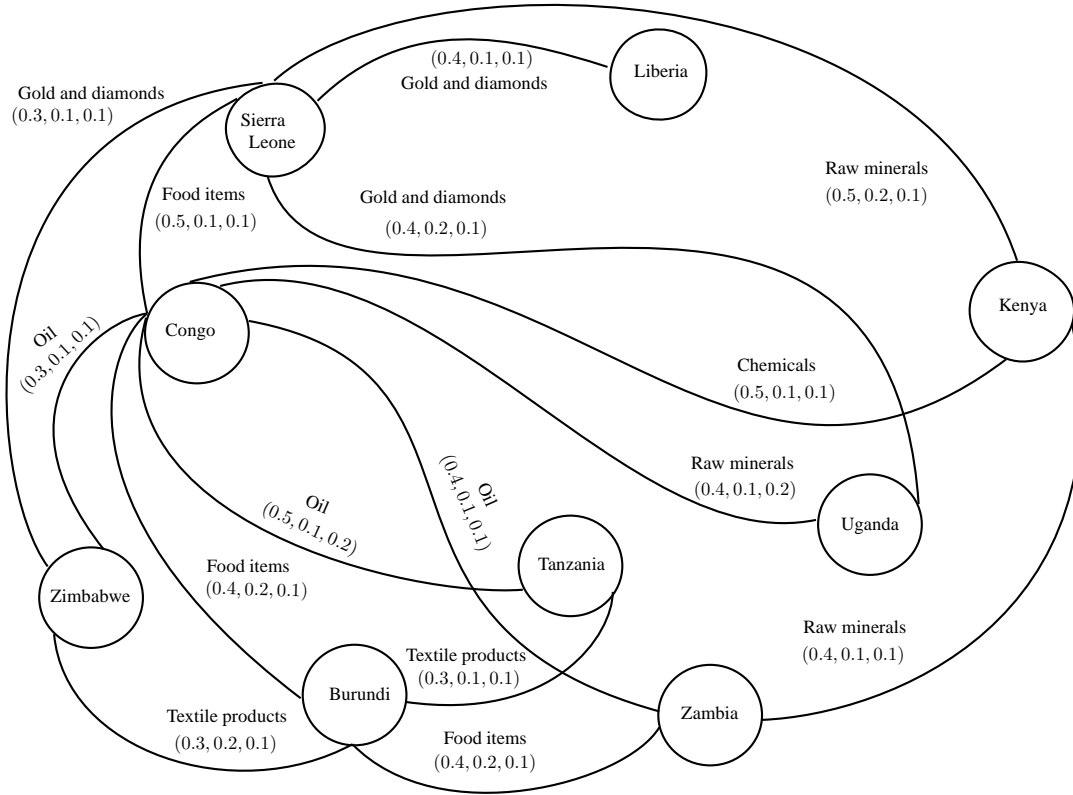
Figure 3.1: SVINSGS indicating eminent trade between any two poor countries

Every edge of this SVINSGS demonstrates the prominent trade between two poor countries, for example prominent trade between Congo and Zambia is Oil, its T, F and I values are 0.4, 0.1 and 0.1, respectively. According to these values, despite of poverty, circumstances of Congo and Zambia are 40% favorable for oil trade, 10% are unfavorable, and 10% are uncertain, that is, sometimes they may be favorable and sometimes unfavorable. We can observe that Congo is vertex with highest vertex degree for relation oil and Sierra Leone is vertex with highest vertex degree for relation gold and diamonds. That is, among these nine poor countries, Congo is most favorable for oil trade, and Sierra Leone is most favorable for trade of gold and diamonds. This SVINSGS will be useful for those investors, who are interested to invest in these nine poor countries. For example an investor can invest in oil in Congo. And if some-one wants to invest in gold and diamonds, this SVINSGS will help him that Sierra Leone is most favorable.

A big advantage of this SVINSGS is that United Nations, IMF, World Bank, and rich countries can be aware of the fact that in which fields of trade, these poor countries are trying to be better and can be helped to make their economic conditions better. Moreover, SVINSGS of poor countries can be very beneficial for them, it may increase trade as well as foreign aid and economic help from the World, and can present their better aspects before the World.

We now explain general procedure of this application by following algorithm.
Algorithm:

Step 1. Input a vertex set $P = \{C_1, C_2, \ldots, C_n\}$ and a SVINS set O defined on set P.

Step 2. Input SVINS set of trade of any vertex with all other vertices and calculate T, F, and I of each pair of vertices using, $T(C_iC_j) \leq \min(T(C_i), T(C_j))$, $F(C_iC_j) \leq \max(F(C_i), F(C_j))$, $I(C_iC_j) \leq \min(I(C_i), I(C_j))$.

Step 3. Repeat Step 2 for each vertex in set P.

Step 4. Define relations $P_1, P_2, \ldots, P_n$ on the set P such that, $(P, P_1, P_2, \ldots, P_n)$ is a GRS.

Step 5. Consider an element of that relation, for which its value of T is comparatively high, and its values of F and I are low than other relations.

Step 6. Write down all elements in relations with T, F and I values, corresponding relations $O_1, O_2, \ldots, O_n$ are SVINS sets on $P_1, P_2, P_3, \ldots, P_n$, respectively and $(O, O_1, O_2, \ldots, O_n)$ is a SVINSGS.

4   Conclusions

Fuzzy graphical models are highly utilized in applications of computer science. Especially in database theory, cluster analysis, image capturing, data mining, control theory, neural networks, expert systems and artificial intelligence. In this research paper, we have introduced certain operations on single-valued intuitionistic neutrosophic graph structures. We have discussed a novel and worthwhile real-life application of single-valued intuitionistic neutrosophic graph structure in decision-making. We have intensions to generalize our concepts to (1) Applications of SVINS soft GRSs in decision-making (2) Applications of SVINS rough fuzzy GRSs in decision-making, (3) Applications of SVINS fuzzy soft GRSs in decision-making, and (4) Applications of SVINS rough fuzzy soft GRSs in decision-making.

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