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A neutrosophic set-based computational model for a time-dependent decision-support system with multi-attribute criteria

Andy St. Fort[1] *Edwin K. P. Chong*[1]

[1] *Department of Systems Engineering, Colorado State University, Fort Collins, Colorado 80523, U.S.A.*
* E-mail: astfort@rams.colostate.edu

Abstract: We present a neutrosophic set-based model for a time-dependent decision-support system (DSS) with multi-attribute criteria decision-making. Such a DSS includes multiple conflicting objectives, having strategies spanning over several discrete time periods. In this paper, we utilize the concept of neutrosophic sets and some of its operations to develop a computational model that captures decision trees with various imprecise preferences for a time-dependent DSS. Given a time-dependent DSS with N objectives spanning over discrete time periods t ranging from t_0 to t_n, we use a set of m attributes, denoted by variables $a_1, \ldots, a_m$, where each variable a_{k_t} $(k = 1, \ldots, m)$, for each $t \in [t_0, t_n]$, is described by a triplet variable $x_k(\tau_{k_t}, i_{k_t}, f_{k_t})$, where the terms τ_{k_t}, i_{k_t}, and f_{k_t} represent degrees of truthfulness membership, indeterminacy membership, and falsity membership for attribute a_{k_t} at time t, respectively. We then define a set of m time-dependent vectors of imprecise consequences S_q corresponding to a set of strategies derived from the membership of each attribute a_{k_t}. We provide an example to illustrate our model. For each time t, we normalize the set of imprecise consequences to define the weighted values for each attribute. We proceed with an interpretation and a sensitivity analysis of the normalized imprecise consequences and derive a ranking process of strategies at each time t. As a result, the model presents the decision maker with a set of strategies ranked based on the neutrosophic values of each corresponding attribute at each time t.

1 Introduction

1.1 Computational modeling in decision-support systems

The challenges of decision-making in complex problems that are rapidly changing and not easily specified in advance have led to the development of model-driven decision-support systems (DSSs) [1]. Model-driven DSSs are complex systems in which a set of specific required data (or attributes) are carefully studied and analyzed to develop multiple sets of strategies, which allows the decision maker to select the most efficient set of strategies that achieves a predefined set of objectives. Model-driven DSSs have evolved over the past three decades from simple model-oriented systems to advanced multi-function entities [2]. This stems from the continuous evolution of the necessary computational modeling required to solve most of these challenges. As a result, the growing need for decision support in such problems has shaped most DSS models to be robustly designed specifically to [3] [4] [5]:

1. facilitate dynamic decision processes through the use of a series of attributes to develop a set of strategies.
(a) Interdependency between each of the attributes and their consequent strategies is accounted for and supported.
(b) Each strategy is then capable of affecting future attributes and/or strategies at some later time.
2. support rather than just automate decision-making.
(a) Some DSS models will make decisions based on predefined rules in the environment where the DSS operates. However, in the most complex situations, many of those rules may not be as relevant as originally thought. Thus, the best DSS models are also capable of presenting the decision maker with relevant information about each strategy and how it aligns with the corresponding objective(s). In the end, the decision maker will select the best set of strategies.
3. be able to respond quickly to the changing needs of decision makers.

(a) DSS models must support a body of knowledge for the DSS [4] [5] that is capable of record keeping, presenting information on an ad hoc basis in both standardized and customized reports, selecting a desired subset of stored strategies for either presentation or for deriving new strategies, and interacting directly with a decision maker in such a way that the user has a flexible choice and sequence of knowledge-management activities [5] [6].

In light of the stated specifications above, efficient model-driven DSSs serve as critical tools for the decision maker, in that they facilitate them with the ability to select the best strategies based on any given set of attributes in the present, foresee consequences in the future, and present ad hoc information on each strategy's alignment with each of the given objectives at any given time. Moreover, in more complex situations (e.g., when the problem contains multiple conflicting objectives spanning over several discrete time periods t ($t \in [t_0, t_n]$) and a predefined set of attributes along with their respective weights and uncertainty are allowed to change over time), model-driven DSSs have been developed to help identify optimal strategies by using *interactive multi-objective simulated annealing* [1] or *imprecise multi-attribute additive modeling* [7]. In both cases, each attribute is assigned an absolute weight value so that at each point in time, the decision maker is able to evaluate its importance or preference over other attributes at that point in time. In contrast, each attribute can also be set to a different weight value at each time t ($t \in [t_0, t_n]$) as more data become available [8]. Either way, it becomes very difficult to assess each attribute (by the decision maker's preference, the attribute's importance, or the attribute's relevance, at time t) when presented with a sufficiently large number of attributes or time units within the $[t_0, t_n]$ interval.

Consequently, since most multi-attribute DSS models use a finite but large number of attributes (time-dependent or not), we approach this problem using a neutrosophic set-based computational model, as described in more detail in Section 2. In the model, rather than

restricting each individual attribute to a weighting value, we first define three categories of membership (truthfulness τ, indeterminacy i, and falsity f). Then, we assign each attribute a time-dependent triplet variable $x(\tau_t, i_t, f_t)$, denoting whether that attribute holds true, indeterminate, or false at time t. Finally, a strategy is developed based on the membership of that attribute at time t. This allows us to only specify what the membership of the attribute is at time t, and therefore enables us to bypass the need to evaluate each attribute individually by preference, importance, or relevance, at each time t, as mentioned above. Such a model is a neutrosophic set-based model, derived from the theory of neutrosophy. Subsection 1.2 highlights the main properties of neutrosophic logic and derived neutrosophic sets necessary for the development of this model.

1.2 Neutrosophic sets and neutrosophy

Neutrosophic sets derive from *neutrosophy*. Neutrosophy and neutrosophic sets were both introduced by Smarandache in 1995 [9] [10] as a generalization of intuitionistic fuzzy logic [11] [12] and sets, respectively. Smarandache defined neutrosophy as the study of origins, nature, and scope of neutralities, as well as their interactions with different ideational spectra [9]. In other words, neutrosophy is the study of ideas and notions that are not true, nor false, but are said to be between true and false, neutral, indeterminate, unclear, vague, ambiguous, incomplete, contradictory, and so forth [13]. Thus, we describe a neutrosophic set as one where each element of the universe of discourse $\mathbb{U}$ has a degree of truth, indeterminacy, and falsity, respectively. The degrees of truth, indeterminacy, and falsity are called *neutrosophic components*. Each neutrosophic component lies in a non-standard, *infinitesimal unit (hyperreal) interval* [14], denoted $]^-0, 1^+[$. Thus, unlike intuitionistic fuzzy sets, there is no constraint between the degree of truth, the degree of indeterminacy, and the degree of falsity in neutrosophic sets. Moreover, in neutrosophic sets, the sum of the scalar neutrosophic components (denoted $n_{n.s.}$) does not necessarily equal to 1. For example, assuming that $\tau, i, f \in]^-0, 1^+[$ are the degrees of truthfulness, indeterminacy, and falsity, respectively, the sum $n_{n.s.} = \tau + i + f$ can be any number between 0 and 3. This leads to the two main distinctions between neutrosophic sets and intuitionistic fuzzy sets, which are (a) in neutrosophic sets, $n_{n.s.}$ can be any number in the range 0 and 3 in order to allow for the characterization of incomplete information while for intuitionistic fuzzy sets, $n_{n.s.}$ is exactly equal to 1; and (b) in neutrosophic sets, we use the non-standard interval $]^-0, 1^+[$ for each of the neutrosophic components to differentiate between absolute membership (denoted by 1^+) and relative membership (denoted by 1), while the standard interval $[0, 1]$ is used in intuitionistic fuzzy sets [9]. Since decision-making involves the analysis of a finite set of alternatives described in terms of evaluative criteria, neutrosophic sets can be useful in the development of DSS models. To this end, we define the general concepts and operations on neutrosophic sets.

1.2.1 General concepts of neutrosophic sets:
We now introduce precise definitions of concepts, terminology, and notation in neutrosophic set theory. These are standard and follow those of Smarandache [10] [13].

Definition 1.1. *(Neutrosophic set) [10] Let A be a subset of the universe of discourse $\mathbb{U}$. We say that A is a neutrosophic set if there are three functions, $T_A : \mathbb{U} \to]^-0, 1^+[$, $I_A : \mathbb{U} \to]^-0, 1^+[$, and $F_A : \mathbb{U} \to]^-0, 1^+[$, that assign the truthfulness, indeterminacy, and falsity values of each $x \in \mathbb{U}$. In other words, to say that A is a neutrosophic set is to say that for each $x \in \mathbb{U}$, there are three numbers: $\tau = T_A(x)$, $i = I_A(x)$, and $f = F_A(x)$. To simplify, we simply write $x(\tau, i, f) \in A$.*

We caution the reader not to misinterpret the notation $x(\tau, i, f)$ to denote a *function* of τ, i, and f. Instead, given $x \in \mathbb{U}$, we write the parenthesized triple (τ, i, f) next to it to indicate that these are the neutrosophic components of x (with respect to A, which must be included as part of the notation $x(\tau, i, f) \in A$).

Definition 1.2. *(Complement) [9] Given a neutrosophic set A, we define the complement of A, denoted A^c, as the neutrosophic set with the property that $x(1 - \tau_A, 1 - i_A, 1 - f_A) \in A^c$ if and only if $x(\tau_A, i_A, f_A) \in A$.*

Definition 1.3. *(Containment) Given neutrosophic sets A_1 and A_2, we write $A_1 \subseteq A_2$ if for all $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ and $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$,*

$$\tau_{A_1} \leq \tau_{A_2}; \tag{1}$$

$$f_{A_1} \geq f_{A_2}. \tag{2}$$

Definition 1.4. *(Union) Given neutrosophic sets A_1 and A_2, for all $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ and $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$, the neutrosophic components of x with respect to the union $A_3 = A_1 \cup A_2$ are defined by*

$$\tau_{A_3} = \tau_{A_1} + \tau_{A_2} - \tau_{A_1} \times \tau_{A_2}; \tag{3}$$

$$i_{A_3} = i_{A_1} + i_{A_2} - i_{A_1} \times i_{A_2}; \tag{4}$$

$$f_{A_3} = f_{A_1} + f_{A_2} - f_{A_1} \times f_{A_2}. \tag{5}$$

Definition 1.5. *(Intersection) Given neutrosophic sets A_1 and A_2, for all $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ and $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$, the neutrosophic components of x with respect to the intersection $A_3 = A_1 \cap A_2$ are defined by*

$$\tau_{A_3} = \tau_{A_1} \times \tau_{A_2}; \tag{6}$$

$$i_{A_3} = i_{A_1} \times i_{A_2}; \tag{7}$$

$$f_{A_3} = f_{A_1} \times f_{A_2}. \tag{8}$$

Definition 1.6. *(Single-valued neutrosophic set) [9] [15] Let $U \subset \mathbb{U}$ be a space of points (or objects) with a generic element x. Given a neutrosophic set A, we say that A is a single-valued neutrosophic set (SVNS), denoted $A(x) = \{\langle x : \tau, i, f \rangle, x \in U\}$, if for all $x(\tau, i, f) \in A$, τ, i, and f are all in the standard interval $[0, 1]$.*

For a SVNS, there is no restriction on the sum $n_{n.s.}$, which may be as low as 0 and as high as 3. Moreover, we caution the reader not to confuse the notation $x(\tau_A, i_A, f_A) \in A$ with $A(x) = \{\langle x : \tau, i, f \rangle, x \in U\}$. The former is the standard representation of a given neutrosophic set A while the latter denotes that A is a SVNS [10] (see Definitions 1.1 and 1.6).

1.2.2 Set-theoretic operations on SVNSs:
The following definitions highlight the set-theoretic operations on SVNSs.

Definition 1.7. *[15] Given a SVNS $A(x) = \{\langle x : \tau_A, i_A, f_A \rangle, x \in U\}$, we define the following:*

1. *The complement of A, denoted A^c, is given by*

$$A^c(x) = \{\langle x : f_A, 1 - i_A, \tau_A \rangle, x \in U\}. \tag{9}$$

2. *For $\lambda > 0$, we have*

$$\lambda \times A(x) = \{\langle x : 1 - (1 - \tau_A)^\lambda, i_A^\lambda, f_A^\lambda \rangle, x \in U\}; \tag{10}$$

$$A^\lambda(x) = \{\langle x : \tau_A^\lambda, 1 - (1 - i_A)^\lambda, 1 - (1 - f_A)^\lambda \rangle,$$
$$x \in U\}. \tag{11}$$

Definition 1.8. *[9] [15] [16] Given two SVNSs $A_1(x) = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1} \rangle, x \in U\}$ and $A_2(x) = \{\langle x : \tau_{A_2}, i_{A_2}, f_{A_2} \rangle, x \in U\}$, we define the following:*

1. *$A_1 \subseteq A_2$, if and only if*

$$\tau_{A_1} \leq \tau_{A_2}, i_{A_1} \geq i_{A_2}, f_{A_1} \geq f_{A_2}. \tag{12}$$

2. $A_1 = A_2$, *if and only if*

$$\tau_{A_1} = \tau_{A_2}, i_{A_1} = i_{A_2}, f_{A_1} = f_{A_2}. \qquad (13)$$

3. $A_3 = A_1 \cup A_2$ *is defined by*

$$A_3(x) = \big\{\langle x : \max(\tau_{A_1}, \tau_{A_2}), \min(i_{A_1}, i_{A_2}),$$
$$\min(f_{A_1}, f_{A_2})\rangle, x \in U\big\}. \qquad (14)$$

4. $A_3 = A_1 \cap A_2$ *is defined by*

$$A_3(x) = \big\{\langle x : \min(\tau_{A_1}, \tau_{A_2}), \max(i_{A_1}, i_{A_2}),$$
$$\max(f_{A_1}, f_{A_2})\rangle, x \in U\big\}. \qquad (15)$$

5. $A_3 = A_1 + A_2$ *is defined by*

$$A_3(x) = \big\{\langle x : \tau_{A_1} + \tau_{A_2} - \tau_{A_1}\tau_{A_2}, i_{A_1}i_{A_2},$$
$$f_{A_1}f_{A_2}\rangle, x \in U\big\}. \qquad (16)$$

6. $A_3 = A_1 \times A_2$ *is defined by*

$$A_3(x) = \big\{\langle x : \tau_{A_1}\tau_{A_2}, i_{A_1} + i_{A_2} - i_{A_1}i_{A_2},$$
$$f_{A_1} + f_{A_2} - f_{A_1}f_{A_2}\rangle, x \in U\big\}. \qquad (17)$$

Definition 1.9. *[15] Given a SVNS $A(x) = \{\langle x : \tau_A, i_A, f_A\rangle, x \in U\}$, the score function $\sigma_A : U \to [0,1]$, accuracy function $\alpha_A : U \to [-1,1]$, and certainty function $\upsilon_A : U \to [0,1]$ of A are defined by*

$$\sigma_A(x) = \frac{2 + \tau_A - i_A - f_A}{3}; \qquad (18)$$

$$\alpha_A(x) = \tau_A - f_A; \qquad (19)$$

$$\upsilon_A(x) = \tau_A. \qquad (20)$$

Definition 1.10. *[15] Given two SVNSs $A_1(x) = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1}\rangle, x \in U\}$ and $A_2(x) = \{\langle x : \tau_{A_2}, i_{A_2}, f_{A_2}\rangle, x \in U\}$, we say that A_1 ranks higher than A_2, denoted $A_1 > A_2$, if $\tau_{A_1} > \tau_{A_2}$ and $i_{A_1} + f_{A_1} < i_{A_2} + f_{A_2}$. Conversely, we say that A_1 ranks lower than A_2, denoted $A_1 < A_2$, if $\tau_{A_1} < \tau_{A_2}$ and $i_{A_1} + f_{A_1} > i_{A_2} + f_{A_2}$.*

Remark 1. *Given two SVNSs $A_1(x) = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1}\rangle, x \in U\}$ and $A_2(x) = \{\langle x : \tau_{A_2}, i_{A_2}, f_{A_2}\rangle, x \in U\}$, the following hold true:*

1. *If $\sigma_{A_1}(x) > \sigma_{A_2}(x)$, then $A_1 > A_2$.*
2. *If $\sigma_{A_1}(x) = \sigma_{A_2}(x)$ and $\alpha_{A_1}(x) > \alpha_{A_2}(x)$, then $A_1 > A_2$.*
3. *If $\sigma_{A_1}(x) = \sigma_{A_2}(x)$, $\alpha_{A_1}(x) = \alpha_{A_2}(x)$, and $\upsilon_{A_1}(x) > \upsilon_{A_2}(x)$, then $A_1 > A_2$.*
4. *If $\sigma_{A_1}(x) = \sigma_{A_2}(x)$, $\alpha_{A_1}(x) = \alpha_{A_2}(x)$, and $\upsilon_{A_1}(x) = \upsilon_{A_2}(x)$, then $A_1 = A_2$.*

We provide the following explanations for the claims in Remark 1:

1. Using equation (18), let $i_{A_1} + f_{A_1} = b_1$ for A_1 and $i_{A_2} + f_{A_2} = b_2$ for A_2, $\sigma_{A_1}(x) > \sigma_{A_2}(x)$ means that $\tau_{A_1} - b_1 > \tau_{A_2} - b_2$. We agree that $\tau_{A_1} \geq \tau_{A_2}$ and $b_1 \leq b_2$. It is easy to see that at a minimum, $A_1 \geq A_2$.
2. Using equation (19), $\alpha_{A_1}(x) > \alpha_{A_2}(x)$ also means that $\tau_{A_1} - f_{A_1} > \tau_{A_2} - f_{A_2}$. Thus, $\tau_{A_1} \geq \tau_{A_2}$ and $f_{A_1} \leq f_{A_2}$ leads to $A_1 \geq A_2$, at least.
3. Letting $i_{A_1} + f_{A_1} = b_1$ for A_1 and $i_{A_2} + f_{A_2} = b_2$ for A_2, we know that $b_1 = b_2$, and $\tau_{A_1} - f_{A_1} = \tau_{A_2} - f_{A_2}$. However, with $\upsilon_{A_1}(x) > \upsilon_{A_2}(x)$, then $\tau_{A_1} > \tau_{A_2}$. Thus, $A_1 > A_2$.

4. Letting $i_{A_1} + f_{A_1} = b_1$ for A_1 and $i_{A_2} + f_{A_2} = b_2$ for A_2, having $b_1 = b_2$, $\tau_{A_1} - f_{A_1} = \tau_{A_2} - f_{A_2}$, and $\tau_{A_1} = \tau_{A_2}$, then $A_1 = A_2$ (see equation (13)).

Remark 2. *The zero set, denoted $0_\mathbb{N}$, is defined by $0_\mathbb{N}(x) = \{\langle x : 0, 1, 1\rangle, x \in U\}$. For all $x \in U$, $\sigma_{0_\mathbb{N}}(x) = 0$, $\alpha_{0_\mathbb{N}}(x) = -1$, and $\upsilon_{0_\mathbb{N}}(x) = 0$.*

The claim in Remark 2 is easy to show (see Definition 1.9).

Definition 1.11. *(Truth- and falsity-favorite) [9] Given a SVNS $A_1(x) = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1}\rangle, x \in U\}$, the SVNS A_2 is a*

1. *truth-favorite of A_1 and is denoted $A_2 = \Delta A_1$, if*

$$A_2(x) = \big\{\langle x : \min(\tau_{A_1} + i_{A_1}, 1), 0, f_{A_1}\rangle, x \in U\big\}. \qquad (21)$$

2. *falsity-favorite of A_1 and is denoted $A_2 = \nabla A_1$, if*

$$A_2(x) = \big\{\langle x : \tau_{A_1}, 0, \min(i_{A_1} + f_{A_1}, 1)\rangle, x \in U\big\}. \qquad (22)$$

Remark 3. *The complement of the zero set $0_\mathbb{N}$, denoted $0_\mathbb{N}^c$, is defined by $0_\mathbb{N}^c(x) = \{\langle x : 1, 0, 0\rangle, x \in U\}$. Thus, $0_\mathbb{N} \subset 0_\mathbb{N}^c$. Furthermore, $0_\mathbb{N}^c$ is the complement of the false-favorite of $0_\mathbb{N}$, with $\sigma_{0_\mathbb{N}^c}(x) = 1$, $\alpha_{0_\mathbb{N}^c}(x) = 1$, and $\upsilon_{0_\mathbb{N}^c}(x) = 1$.*

The claim in Remark 3 is easy to show (see Definitions 1.8, 1.9, and 1.11).

Lemma 1.1. *Given a SVNS $A(x) = \{\langle x : \tau_A, i_A, f_A\rangle, x \in U\}$, we have $0_\mathbb{N} \subseteq A$.*

Proof: This claim is a generalization of Remark 3. If A is a zero set, then $A = 0_\mathbb{N}$. Otherwise, for all $x \in U$, it is clear that $0 \leq \tau_A$, $1 \geq i_A$, and $1 \geq f_A$ for all $\tau_A, i_A, f_A \in [0,1]$ (see Definition 1.8, equation (12)). $\qquad \square$

2 Developing a DSS computational model using SVNSs

2.1 Defining objectives and attributes

Using set-theoretic operations on SVNSs, we now describe a model to support a finite number N of *objectives* over discrete time periods ranging from t_0 to t_n. The set of N objectives are a predefined set of decision alternatives well-known to the decision maker. Additionally, each of the time periods represent an *objective level*. That is, at each $t \in [t_0, t_n]$, a specific set of *attributes* are used to develop a course of action (in other words, a *strategy*) to achieve a particular objective and to determine whether that strategy is suitable over those of other objectives at that level. Thus, having discrete time periods in $[t_0, t_n]$ implies having $n + 1$ objective levels. Since the objectives are spanned over t_0 to t_n, the ratio

$$m = \frac{N}{n+1}$$

represents the number of attributes per objective level. The model supports the rearrangement of the given objectives into an *objective tree* [1] with $n + 1$ objective levels matching the time interval $[t_0, t_n]$, as illustrated in Figure 1. Previous studies have been able to determine that an objective's importance can change over time based on how the decision maker perceives each objective and its alignment with the strategy developed for each attribute [1]; thus, the model also needs to take this finding into account. As a result, the model is designed to support m time-dependent attributes, denoted $a_1, \ldots, a_m$, spanning over each $t \in [t_0, t_n]$ (Figure 1). That
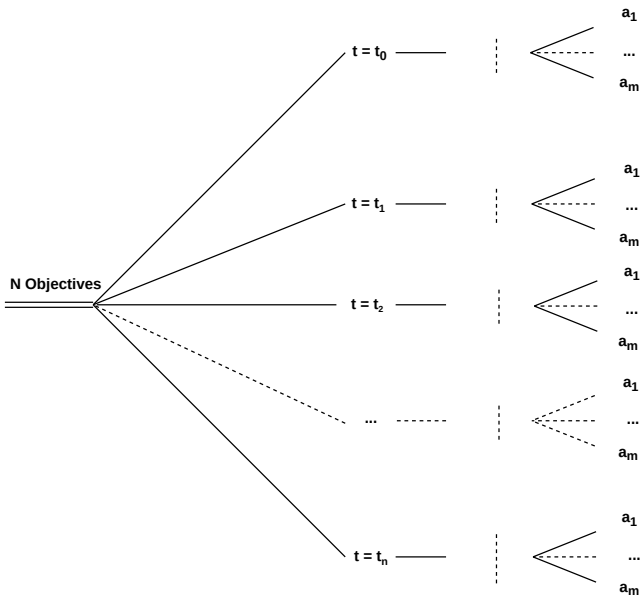
Fig. 1: Objective tree including time periods [1].

is, each a_{k_t}, with $k \in [1, m]$, is a tuple:

$$a_{k_t} = \left\{ a_{k_{t_0}}, \ldots, a_{k_{t_n}} \right\}. \tag{23}$$

Each attribute's value at time t, i.e., a_{k_t}, is determined based on the neutrosophic values assigned to that attribute. That is, we bypass the need to determine the relevance of attribute a_{k_t} at time t beforehand. The model requires specifying only whether a_{k_t} is true, indeterminate, or false at time t. Then, we define each a_{k_t} as a SVNS (see Definition 2.1). Ultimately, for this model, the subset $U \subset \mathbb{U}$ defined earlier as a space of points or objects (from Definition 1.6) becomes the Cartesian product $[1, m] \times [t_0, t_n]$ with a generic element x, and is reflected in equation (24). To this end, the first definition for the model is as follows:

Definition 2.1. *Given a non-empty set of N predefined objectives and m time-dependent attributes, all spanning over $[t_0, t_n]$, for each time t and for each k, the attribute a_{k_t} is a SVNS and is defined by*

$$a_{k_t}(x) = \left\{ \left\langle x : \tau_{k_t}, i_{k_t}, f_{k_t} \right\rangle, x \in [1, m] \times [t_0, t_n] \right\}, \tag{24}$$

where $\tau_{k_t}, i_{k_t}, f_{k_t} \in [0, 1]$, $m \in \mathbb{N}^*$, $k \in [1, m]$, and $t \in [t_0, t_n]$.

Lemma 2.1. *For $k \in [1, m]$ and $t \in [t_0, t_n]$, a_{k_t} is a non-zero set.*

Proof: We provide the following explanations for this claim:

1. For $k \in [1, m]$ and all t, $0_\mathbb{N} \subseteq a_{k_t}$ (see Lemma 1.1).
2. Let $x_{\text{val}} \in U$, where $U = [1, m] \times [t_0, t_n]$, for which we have two non-empty SVNSs $a_{\text{val}_1}(x_{\text{val}}) = \left\{ \left\langle x_{\text{val}} : \tau_{\text{val}_1}, i_{\text{val}_1}, f_{\text{val}_1} \right\rangle, x_{\text{val}} \in U \right\} \subset a_{k_t}$ and $a_{\text{val}_2}(x_{\text{val}}) = \left\{ \left\langle x_{\text{val}} : \tau_{\text{val}_2}, i_{\text{val}_2}, f_{\text{val}_2} \right\rangle, x_{\text{val}} \in U \right\} \subset a_{k_t}$. If there exist $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 \times a_{\text{val}_1} + \lambda_2 \times a_{\text{val}_2} \subset a_{k_t}$, then a_{k_t} is a non-zero

set. From Definition 1.7, we know that

$$\lambda_1 \times a_{\text{val}_1}(x_{\text{val}}) = \left\{ \left\langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_1})^{\lambda_1}, i_{\text{val}_1}^{\lambda_1}, \right. \right.$$
$$\left. \left. f_{\text{val}_1}^{\lambda_1} \right\rangle, x_{\text{val}} \in U \right\}; \tag{25}$$

$$\lambda_2 \times a_{\text{val}_2}(x_{\text{val}}) = \left\{ \left\langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_2})^{\lambda_2}, i_{\text{val}_2}^{\lambda_2}, \right. \right.$$
$$\left. \left. f_{\text{val}_2}^{\lambda_2} \right\rangle, x_{\text{val}} \in U \right\}. \tag{26}$$

Thus, let $a_{\text{val}} = \lambda_1 \times a_{\text{val}_1} + \lambda_2 \times a_{\text{val}_2}$. We have

$$a_{\text{val}}(x_{\text{val}}) = \left\{ \left\langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_1})^{\lambda_1} (1 - \tau_{\text{val}_2})^{\lambda_2}, \right. \right.$$
$$\left. \left. i_{\text{val}_1}^{\lambda_1} i_{\text{val}_2}^{\lambda_2}, f_{\text{val}_1}^{\lambda_1} f_{\text{val}_2}^{\lambda_2} \right\rangle, x_{\text{val}} \in U \right\}. \tag{27}$$

We know that $\tau_{\text{val}_1}, i_{\text{val}_1}, f_{\text{val}_1}, \tau_{\text{val}_2}, i_{\text{val}_2}, f_{\text{val}_2} \in [0, 1]$, thus for any $\lambda_1, \lambda_2 > 0$, we have $0 \leq i_{\text{val}_1}^{\lambda_1} i_{\text{val}_2}^{\lambda_2} \leq 1$ and $0 \leq f_{\text{val}_1}^{\lambda_1} f_{\text{val}_2}^{\lambda_2} \leq 1$. Furthermore, $1 - \tau_{\text{val}_1} \leq 1$ implies that $(1 - \tau_{\text{val}_1})^{\lambda_1} \leq 1$. The same applies for $(1 - \tau_{\text{val}_2})^{\lambda_2}$. As a result, $0 \leq 1 - (1 - \tau_{\text{val}_1})^{\lambda_1} (1 - \tau_{\text{val}_2})^{\lambda_2} \leq 1$ for any $\lambda_1, \lambda_2 > 0$. Now, since with $x_{\text{val}} \in U$, and $a_{\text{val}_1}, a_{\text{val}_2} \subset a_{k_t}$, then $a_{\text{val}} \subset a_{k_t}$ for all $x_{\text{val}} \in U$.
3. Let $a_{\text{val}_1} \cap a_{\text{val}_2} \subset a_{k_t}$. If $a_{\text{val}_1} \cap a_{\text{val}_2} = 0_\mathbb{N}$, then condition 1 applies. Otherwise,

$$[a_{\text{val}_1} \cap a_{\text{val}_2}](x_{\text{val}}) = \left\{ \left\langle x_{\text{val}} : \min(\tau_{\text{val}_1}, \tau_{\text{val}_2}), \right. \right.$$
$$\left. \left. \max(i_{\text{val}_1}, i_{\text{val}_2}), \max(f_{\text{val}_1}, f_{\text{val}_2}) \right\rangle, x_{\text{val}} \in U \right\}. \tag{28}$$

Since $a_{\text{val}_1}, a_{\text{val}_2}$ are non-empty sets, then $a_{\text{val}_1} \cap a_{\text{val}_2}$ from equation (28) is a non-empty set, stemming from the fact that the values $\min(\tau_{\text{val}_1}, \tau_{\text{val}_2})$, $\max(i_{\text{val}_1}, i_{\text{val}_2})$, and $\max(f_{\text{val}_1}, f_{\text{val}_2})$ exist. Then, $a_{\text{val}_1} \cap a_{\text{val}_2} \subset a_{k_t}$ implies that a_{k_t} is a non-empty set.

As a result, for all k and t, a_{k_t} is a non-zero set. $\square$

In decision-making, to achieve defined objectives, one or more strategies are developed. Those strategies are developed using attributes that relate each objective with defined checkpoints such as when to take a course of action and when to assess whether the objective(s) has/have been met. We apply this idea in the model. Each SVNS a_{k_t}'s neutrosophic component values are used to determine a set of strategies. However, as we often encounter, strategies can be difficult to assess quantitatively. Therefore, the set of strategies is characterized by a set of *strategy consequences* that allows us to utilize the neutrosophic values from each a_{k_t} and derive a vector of intervals containing time-dependent *consequence values* for each a_{k_t}. We describe the obtained time-dependent consequence values as *imprecise consequences*. This stems from the fact that the set of strategies developed are based on neutrosophic values that encompass unclear, ambiguous, or incomplete knowledge of each attribute at time t. The decision maker will then assess each attribute based on the imprecise consequence for that attribute. Last, it is worth noting that this model does not manipulate or make changes to the objectives; any manipulation/assessment other than rearrangement of attributes into an objective tree is left to the decision maker.

From Definition 2.1 and Lemma 2.1, it is possible to rank attributes using the score function, compare attributes using the accuracy function, and determine the likelihood of an attribute using the certainty function (see Definition 1.9). In doing so, we are also able to derive strategy consequences from attributes. The next subsection introduces the definitions for strategies and their consequences.

2.2 Defining consequences of strategies

Definition 2.2. *Given a non-empty set of N predefined objectives and m time-dependent attributes represented by the SVNS $a_{k_t}(x) = \left\{ \langle x : \tau_{k_t}, i_{k_t}, f_{k_t} \rangle, x \in [1, m] \times [t_0, t_n] \right\}$, let S be the set of available strategies derived from all a_{k_t}. At time t, the set S_q of imprecise consequences of such strategies is a stream defined by a vector of imprecise consequences, denoted $s_{q_k}(t)$. Thus, we write the set of imprecise consequences as*

$$S_q = \left\{ s_{q_1}(t), \ldots, s_{q_m}(t) \right\}; \qquad (29)$$

with each $s_{q_k}(t)$, $k \in [1, m]$, defined by

$$s_{q_k}(t) \in \left[s_k^L(t), s_k^U(t) \right], \qquad (30)$$

where $s_k^L(t)$ and $s_k^U(t)$ are (respectively) the lower and upper endpoints of the imprecise consequence interval for attribute a_{k_t}.

Assumption 2.1. *For all a_{k_t}, $S_q \subseteq S$.*

Assumption 2.1 is critical for this model. From equation (30), we can see that $s_{q_k}(t)$ is a value from the interval $\left[s_k^L(t), s_k^U(t) \right]$. Thus, for each a_{k_t}, S contains all possible values within $\left[s_k^L(t), s_k^U(t) \right]$. Hence, the description of S being the set of all available strategies for all a_{k_t}.

Assumption 2.2. *For all a_{k_t}, there is a continuous distribution between $s_k^L(t)$ and $s_k^U(t)$ endpoints.*

We will discuss the need for Assumption 2.2 later in this subsection.

Definition 2.3. *Given a non-empty set of N predefined objectives and m time-dependent attributes represented by the SVNS $a_{k_t}(x) = \left\{ \langle x : \tau_{k_t}, i_{k_t}, f_{k_t} \rangle, x \in [1, m] \times [t_0, t_n] \right\}$, let S be the derived set of available strategies. Each imprecise consequence interval $\left[s_k^L(t), s_k^U(t) \right]$ for a_{k_t} is restricted to the following conditions:*

1. *If $i_{k_t} = 0$, then*

$$s_{q_k}(t) = s_k^L(t) = s_k^U(t) = \tau_{k_t}. \qquad (31)$$

2. *If $i_{k_t} > 0$, then*

$$s_k^L(t) = \min \left(\frac{2 + \tau_{k_t} - i_{k_t}^* - f_{k_t}}{3} \right), \qquad (32)$$

$$s_k^U(t) = \max \left(\frac{2 + \tau_{k_t} - i_{k_t}^* - f_{k_t}}{3} \right), \qquad (33)$$

and

$$s_k^L(t) \leq s_{q_k}(t) \leq s_k^U(t), \qquad (34)$$

where $i_{k_t}^* = \{ i_{k_t}, 1 - i_{k_t} \}$.

In Definition 2.3, we put an emphasis on the indeterminacy of a_{k_t}. This stems from the fact that once we are able to precisely characterize attribute a_{k_t}, i.e., $i_{k_t} = 0$, we agree that its certainty is based on the value of τ_{k_t}, i.e., $v_{a_{k_t}}(x) = \tau_{k_t}$. We describe this scenario (i.e., $i_{k_t} = 0$) as having *precise knowledge* of a_{k_t}. This is also applicable when $\tau_{k_t} = 0$, leading to $s_k^L(t) = s_k^U(t) = 0$, and both endpoints take the lowest possible value of 0. Conversely, with $i_{k_t} = 0$ and $\tau_{k_t} = 1$, then $s_k^L(t) = s_k^U(t) = 1$, and both endpoints take the highest possible value of 1. Now, if $i_{k_t} > 0$, we describe this scenario as having *imprecise knowledge* of a_{k_t}. Therefore, having imprecise knowledge of the nature of attribute a_{k_t} implies that its indeterminacy becomes prevalent, and is used in determining both

endpoints of the imprecise consequence for a_{k_t}. We do so by creating a masked indeterminacy value $i_{k_t}^*$ that takes both the value of i_{k_t} and the value of the complement $1 - i_{k_t}$, and substituting them in the score function of a_{k_t}, as seen in equations (32) and (33).

A special case is where $i_{k_t} = 1 - i_{k_t}$, meaning $i_{k_t} = 0.5$, then $s_k^L(t) = s_k^U(t) = \sigma_{a_{k_t}}(x)$ (see equation (18)). As a result, each $s_{q_k}(t)$ is expected to be within an interval with distinct endpoints, except for when $i_{k_t} \in \{0, 0.5\}$, in which case $s_{q_k}(t) = s_k^L(t) = s_k^U(t)$. We interpret this result as (a) having precise knowledge of a_{k_t} leads to a single strategy consequence that is solely based on the certainty of attribute a_{k_t} at time t, (b) having an imprecise knowledge of a_{k_t} at about 50% leads to a single strategy consequence that takes all neutrosophic components into consideration.

It is worth noting that, when $i_{k_t} \neq 0$, we have an interval of imprecise consequence values as seen in equation (30). This entails that we need some creative way to generate the values between $s_k^L(t)$ and $s_k^U(t)$ and estimate the best $s_{q_k}(t)$ value(s) from the interval for decision-making purposes. Obviously, this model is intended to empower the decision maker with a weighting tool in which attributes' impacts on the current and future strategies are accounted for. By presenting each potential strategy as a direct consequence of a particular attribute in the form of an interval, it is critical to predict which value(s) from the interval are most likely, based on i_{k_t}. Thus, we proceed with the following analysis:

First, it is clear that $0 \leq s_{q_k}(t) \leq 1$ regardless of the value of i_{k_t}. Second, let $i_{k_t}^- = 1 - i_{k_t}$, if $i_{k_t} < 0.5$, we agree that $i_{k_t}^- > i_{k_t}$, then,

$$s_k^L(t) = \frac{2 + \tau_{k_t} - i_{k_t}^- - f_{k_t}}{3}. \qquad (35)$$

Conversely, with $i_{k_t} > 0.5$, we have $i_{k_t}^- < i_{k_t}$, and

$$s_k^U(t) = \frac{2 + \tau_{k_t} - i_{k_t}^- - f_{k_t}}{3}. \qquad (36)$$

Consequently, for all $i_{k_t} \neq 0$, it is safe to say that

$$\frac{2 + \tau_{k_t} - \max(i_{k_t}^-, i_{k_t}) - f_{k_t}}{3} \leq s_{q_k}(t)$$
$$\leq \frac{2 + \tau_{k_t} - \min(i_{k_t}^-, i_{k_t}) - f_{k_t}}{3}. \qquad (37)$$

Naturally, it is a more favorable scenario to have $i_{k_t}^- > i_{k_t}$. We describe such scenario as *favorable indeterminacy*. This stems from the fact that precise knowledge about an attribute at time t is achieved only when $i_{k_t} \to 0$. We also describe the opposite scenario, i.e., $i_{k_t}^- < i_{k_t}$, as *unfavorable indeterminacy*. Therefore, the smaller the indeterminacy, the larger the impact of the attribute's certainty at time t. Moreover, for $i_{k_t} \neq 0.5$, it is clear that there exists a $\zeta_t > 0$ such that $| i_{k_t} - i_{k_t}^- | = \zeta_t$. Then, we provide the following definition and remarks.

Definition 2.4. *Given a non-empty set of N predefined objectives and m time-dependent attributes represented by the SVNS $a_{k_t}(x) = \left\{ \langle x : \tau_{k_t}, i_{k_t}, f_{k_t} \rangle, x \in [1, m] \times [t_0, t_n] \right\}$, let S be the derived set of all available strategies. We define the reverse indeterminate SVNS of a_{k_t}, denoted $a_{k_{rt}}$, as the SVNS with the property that $a_{k_{rt}}(x) = \left\{ \langle x : \tau_{k_t}, 1 - i_{k_t}, f_{k_t} \rangle, x \in [1, m] \times [t_0, t_n] \right\}$. The resulted imprecise consequence interval $s_{k_r}^L(t) \leq s_{q_{k_r}}(t) \leq s_{k_r}^U(t)$ is called the reverse imprecise consequence interval of a_{k_t}.*

Remark 4. *Given a_{k_t} and $a_{k_{rt}}$, let $i_{k_t}^- = 1 - i_{k_t}$ and $i_{k_t} > 0$.*

1. *If $i_{k_t} = i_{k_t}^-$, then $a_{k_t} = a_{k_{rt}}$, and*

$$s_{q_{k_r}}(t) = s_{q_k}(t) = \sigma_{a_{k_t}}(x). \qquad (38)$$

2. If $i_{k_t} > i_{k_t}^-$, then $s_k^L(t) \le s_{q_{k_r}}(t) \le s_{k_r}^U(t)$, where

$$s_k^L(t) = \frac{2 + \tau_{k_t} - i_{k_t} - f_{k_t}}{3}, \qquad (39)$$

$$s_{k_r}^U(t) = \frac{2 + \tau_{k_t} - i_{k_t}^- - f_{k_t}}{3}. \qquad (40)$$

3. If $i_{k_t} < i_{k_t}^-$, then $s_{k_r}^L(t) \le s_{q_k}(t) \le s_k^U(t)$, where

$$s_{k_r}^L(t) = \frac{2 + \tau_{k_t} - i_{k_t}^- - f_{k_t}}{3}, \qquad (41)$$

$$s_k^U(t) = \frac{2 + \tau_{k_t} - i_{k_t} - f_{k_t}}{3}. \qquad (42)$$

The claim in Remark 4 is easy to show (see Definitions 1.9, 2.3, and 2.4).

Lemma 2.2. *For distinct $k, l \in [1, m]$, let $\left[s_k^L(t), s_k^U(t) \right] \subseteq \left[s_l^L(t), s_l^U(t) \right]$. If $i_{k_t} = \min(i_{k_t}, i_{k_t}^-)$ and $i_{l_t} = \min(i_{l_t}, i_{l_t}^-)$, then $a_{k_t} \le a_{l_t}$.*

Proof: We provide the following explanation for this claim. We are aware that each interval is a continuous distribution (See Assumption 2.2), but we also know that $\left[s_k^L(t), s_k^U(t) \right] \subseteq \left[s_l^L(t), s_l^U(t) \right]$ is equivalent to $s_l^L(t) \le s_k^L(t) \le s_k^U(t) \le s_l^U(t)$. That said, from equation (37), we agree that

$$\frac{2 + \tau_{l_t} - \max(i_{l_t}^-, i_{l_t}) - f_{l_t}}{3}$$
$$\le \frac{2 + \tau_{k_t} - \max(i_{k_t}^-, i_{k_t}) - f_{k_t}}{3}, \qquad (43)$$

and

$$\frac{2 + \tau_{k_t} - \min(i_{k_t}^-, i_{k_t}) - f_{k_t}}{3}$$
$$\le \frac{2 + \tau_{l_t} - \min(i_{l_t}^-, i_{l_t}) - f_{l_t}}{3}. \qquad (44)$$

Then, we can deduce that

$$\mid i_{k_t} - i_{k_t}^- \mid \le \mid i_{l_t} - i_{l_t}^- \mid. \qquad (45)$$

From Remark 4, we know that if

1. $i_{k_t} > i_{k_t}^-$, then $s_k^L(t) \le s_{q_{k_r}}(t) \le s_{k_r}^U(t)$;
2. $i_{k_t} < i_{k_t}^-$, then $s_{k_r}^L(t) \le s_{q_k}(t) \le s_k^U(t)$.

Doing the same for i_{l_t} and $i_{l_t}^-$; when $i_{k_t} = \min(i_{k_t}, i_{k_t}^-)$ and $i_{l_t} = \min(i_{l_t}, i_{l_t}^-)$, we can see that $s_k^U(t) \le s_l^U(t)$. As a result, $a_{k_t} \le a_{l_t}$ (see Remark 1). □

In Assumption 2.2, we claimed that there is a continuous distribution between $s_k^L(t)$ and $s_k^U(t)$ endpoints for each a_{k_t}, thus we agree that the probability that a particular value between $s_k^L(t)$ and $s_k^U(t)$ endpoints is assumed is 0; which is fine, as the intent here is not to handpick a value from that interval. Assuming that the continuous distribution between $s_k^L(t)$ and $s_k^U(t)$ endpoints is uniform, if $s_{q_t}^-$ is the distribution mean, then we can imply that, for any values

$\epsilon, y > 0$,

$$\Pr \left[s_{q_t}^- - \epsilon \le y \le s_{q_t}^- \right] = \Pr \left[s_{q_t}^- \le y \le s_{q_t}^- + \epsilon \right]. \qquad (46)$$

Then, it is easy to see that the probability density function is

$$f(y) = \begin{cases} \dfrac{1}{s_k^U(t) - s_k^L(t)}, & \text{if } s_k^L(t) \le y \le s_k^U(t); \\ \\ 0, & \text{otherwise}; \end{cases} \qquad (47)$$

and that any value within the $\left[\min(i_{k_t}, i_{k_t}^-), \max(i_{k_t}, i_{k_t}^-) \right]$ yields equally probable real y values, with the condition that $s_k^L(t) \le y \le s_k^U(t)$.

Now, we try a more complex distribution. Assuming that the continuous distribution is normal, using the *central limit theorem* (*CLT*) [17], we know that the density of the sum of two or more independent variables within the $s_k^L(t) \le y \le s_k^U(t)$ interval is the convolution of their densities [18]. That is, as we add more independent variables to the sum, the density of the sum tends to converge towards the normal density. If $s_{q_t}^-$ is the distribution mean at time t, let n_s be the number of real values within $\left[s_k^L(t), s_k^U(t) \right]$, we expect $n_s \to \infty$. Thus, the classical CLT states that as n_s gets sufficiently large, the distribution gets close to the normal distribution with mean $s_{q_t}^-$ and variance δ^2. As a result, within each closed set $\left[s_k^L(t), s_k^U(t) \right]$, we want $s_{q_k}(t)$ to be as close to the mean $s_{q_t}^-$ as possible. In the end, each attribute a_{k_t} has a consequence of strategy that is either presented as a single value within $[0, 1]$ or as a continuous distribution in $\left[s_k^L(t), s_k^U(t) \right]$, dependent on whether $i_{k_t} = 0$ or not. This gives the decision maker the freedom to (a) develop a problem-solving approach on approximation and conduct sensitivity analysis as needed, or (b) perform discounting on each attribute over time once more data about each attribute become available and perform further sensitivity (or any other) analysis as needed. Since this model solely relies on current data based on the neutrosophic values of each attribute, we use an example in which we apply (a) and leave (b) for further discussions on this topic.

3 Computing example

3.1 Model example input and computation

We use an example in which we apply the definitions, lemmas, and remarks from Section 2. We use an objective tree over time that consists of $N = 110$ objectives, $n = 11$ objective levels derived from 11 time periods ranging from $t = 0$ to $t = 10$. We are also given 10 attributes, defined by the time-dependent set $a = \{a_1, a_2, \ldots, a_{10}\}$, as seen in Table 1. We describe the time periods as a tuple $t = \{0, 1, \ldots, 10\}$, in order to account for the initial knowledge or characteristics of attributes at time $t = 0$. At time $t = 0$, it is expected that the decision maker's knowledge of each attribute and its relevance to objectives is precise, and therefore there are no indeterminacy and $i_k = 0$, with $k \in [1, 10]$. It is also assumed that each attribute a_{k_t} is a SVNS, therefore each τ_{k_t}, i_{k_t}, and f_{k_t} are in $[0, 1]$. Also, each a_{k_t} is a 11-tuple in the form of equations (23) and (24). All simulations are run using the input values recorded in Tables 1 and 2. Neutrosophic components for each a_{k_t}, with $t \in [1, 10]$, are recorded in Table 2.

We calculate the score, accuracy, and certainty values for each a_{k_t} for all t using equations (18) through (20), as seen in Table 3. Using Definition 2.3 and equations (31) through (33), we then determine the imprecise consequences intervals for each a_{k_t} in Table 4. As expected, since the neutrosophic values used for $t = 0$ have no indeterminacy, i.e., $i_{k_t} = 0$, the imprecise consequences intervals only takes one value, which is the value of τ_{k_t}. In this case, the term "imprecise" is an oxymoron since technically, the consequence for each a_{k_t} is precise at $t = 0$ and refers to the certainty of that a_{k_t}. At times $t > 0$, we have determined the imprecise consequence intervals with

both $s_k^L(t)$ and $s_k^U(t)$ endpoints for each attributes a_{k_t}. From Assumption 2.2, there is a continuous distribution between $s_k^L(t)$ and $s_k^U(t)$. This is also taken into account in our analysis.

3.2 Approximation-based approach

In Definition 2.4, Remark 4, and Lemma 2.2, we introduced the terms *reverse indeterminate SVNSs* and *reverse imprecise consequence intervals* for each attribute when presented with the scenario in which $i_{k_t} > i_{k_t}^-$. This is critical in determining one of the two endpoints in the imprecise intervals for each a_{k_t} at time t when presented with an indeterminacy that is larger than 0.5. We also determined that, for distinct $k, l \in [1, m]$, having $\left[s_k^L(t), s_k^U(t)\right] \subseteq \left[s_l^L(t), s_l^U(t)\right]$ where $i_{k_t} = \min(i_{k_t}, i_{k_t}^-)$ and $i_{l_t} = \min(i_{l_t}, i_{l_t}^-)$ leads to the conclusion that $a_{k_t} \leq a_{l_t}$ from an attribute scoring or ranking standpoint. Depending on the number of objectives and attributes being present, drawing such conclusion can be a tough task for a fairly large number of attributes. Thus, we describe the following approximation-based approach to help determine the most relevant attribute(s) at time t, using equation (48). We assume that each imprecise consequence interval is a normal distribution, and we look at the median of each distribution at time t, denoted $s_{q_k}^-$, as seen in Table 5. Once again, at time $t = 0$, we had both endpoints being equal (See Table 4); therefore, for $t = 0$, $s_{q_k}^- = \tau_{k_t}$. For each t, the $s_{q_k}^-$ values from Table 5 are then normalized to generate each attribute weight value, denoted $s_{q_k}^{\hat{}}$ (See Table 6). The results from Table 6 allow the decision maker to rank each attribute based on the imprecise consequence values at each time t. As a result, to obtain the $s_{q_k}^{\hat{}}$ values in Table 6, at each t, with $m = 10$ and $k \in [1, m]$, we use

$$s_{q_k}^{\hat{}} = \frac{s_{q_k}^-}{\sum\limits_{k=1}^{m} s_{q_k}^-} \tag{48}$$

and

$$\sum_{k=1}^{m} s_{q_k}^{\hat{}} = 1. \tag{49}$$

3.3 Interpretation of results and sensitivity analysis

By corresponding each weight value from Table 6 to its relative attribute, the decision maker can choose to prioritize directly based on these weight values. Additionally, these values can also be used to perform discounting on each attribute with the end goal being the prioritization of attributes at time t. Using discounting, however, is an extra measure for comparing attributes, for the purpose of this paper; it has been left to the decision maker's discretion.

At first glance, it is easy to see that at time $t = 0$, attribute a_5 would be ranked first, followed closely by attribute a_9, then by a_2, a_{10}, a_1, a_7, a_6, a_4, a_3, and a_8, respectively. Doing the same for $t = 1$, the attribute ranking is a_6, a_1, a_2, a_3, a_5, a_{10}, a_7, a_4, a_9, and a_8, respectively. For $t = 2$, we have a_{10}, a_2, a_6, a_1, a_7, a_3, a_4, a_8, a_5, and a_9, in that order. The same process is used to determine the ranking or priority for the same attributes at times $t = 3, \ldots, 10$.

The approximation-based approach using the medians of the imprecise consequence intervals in Subsection 3.2 gives us an outlet in assigning a weighted value to each attribute because it allows us to bypass the need to compare each imprecise consequence interval with another (See Lemma 2.2). That approach, however, does not take into account whether attribute a_{k_t} contains a favorable or unfavorable indeterminacy at time t. This stems from the fact that the median is the closest to the halfway point between $s_k^L(t)$ and $s_k^U(t)$, therefore that approach does not take into account whether $i_{k_t} = \min(i_{k_t}, i_{k_t}^-)$ (favorable indeterminacy) or $i_{k_t} = \max(i_{k_t}, i_{k_t}^-)$ (unfavorable indeterminacy). We know that each $\left[s_k^L(t), s_k^U(t)\right]$ is a continuous distribution, so we define $\epsilon_{k_t} = \mathrm{d}s_{q_k}(t)$ as an arbitrary infinitesimal variation from the median $s_{q_k}(t)$ such that $s_{q_k}^-(t) - \epsilon_{k_t} \leq s_{q_k}^-(t) \leq s_{q_k}^-(t) + \epsilon_{k_t}$, at time t.

Thus, we recompute the weighted values $(s_{q_k}^{\hat{}})$ using

$$s_{q_k}^{\hat{}} = \begin{cases} \dfrac{s_{q_k}^- + \epsilon_{k_t}}{\sum\limits_{k=1}^{m} s_{q_k}^-}, & \text{if } i_{k_t} = \min(i_{k_t}, i_{k_t}^-); \\[3ex] \dfrac{s_{q_k}^- - \epsilon_{k_t}}{\sum\limits_{k=1}^{m} s_{q_k}^-}, & \text{otherwise.} \end{cases} \tag{50}$$

For simplicity, we use the same ϵ_{k_t} throughout $t = 1, \ldots, 10$. However, it is normal to envision a case where the decision maker would choose a different ϵ_{k_t} value as t progresses. Moreover, since at $t = 0$, there is no imprecise consequence interval for any of the attributes (See Table 4), the newly computed values only affect the previous weighted values from Table 6 for $t > 0$. Those newly computed weighted values are then reflected in Table 7 using $\epsilon_{k_t} = 0.1$.

As an interpretation of the results in Table 7, we can see that there are no new attribute weighting or ranking for $t = 0$, as expected. For $t = 1$, we have the attributes in which $i_{k_t} = \min(i_{k_t}, i_{k_t}^-)$ (i.e., a_6, a_1, and a_5, respectively), then followed by those in which $i_{k_t} \neq \min(i_{k_t}, i_{k_t}^-)$, such as a_2, a_3, a_{10}, a_7, a_4, a_9, and a_8, in that order. For $t = 2$, we have a_{10}, a_2, a_6, a_1, a_7, a_3, a_9, a_4, a_8, and a_5, in that order. The same process is repeated to determine the rankings at times $t = 3, \ldots, 10$.

As expected, our first observation is that when $i_{k_t} = \min(i_{k_t}, i_{k_t}^-)$, attributes with the largest τ_{k_t} and smallest i_{k_t} values yield a larger $s_{q_k}^{\hat{}}$ than those that do not. Moreover, we can see that when two or more attributes contain $i_{k_t} \neq \min(i_{k_t}, i_{k_t}^-)$, priority is given to the one(s) with the largest τ_{k_t}. This new ranking aligns more with Lemma 2.2 than the approximation-based approach of just using the normalized weights of the median values of the imprecise consequence intervals reflected in Table 6. For instance, at $t = 4$, a_4 has $i_{k_t} = 0.39$ while i_{k_t} for a_9 is 0.198; we also can see that $[0.711, 0.785] \subseteq [0.669, 0.87]$, and according to both Lemma 2.2 and the $s_{q_k}^{\hat{}}$ values obtained in Table 7, we observe that a_9 would be ranked just ahead of a_4. As a result, where applicable, either Lemma 2.2 or the approach in equation (50) can be used to determine which attribute has the most impactful strategy between any given set of attributes, at a specific time in the future. The challenge, in applying the approach in equation (50), is how to determine which ϵ_{k_t} is best to facilitate prioritizing attributes with favorable indeterminacy over those with unfavorable indeterminacy. We can see that the standard deviations from the imprecise consequence interval medians, at each $t > 0$, are 0.178, 0.163, 0.247, 0.189, 0.179, 0.174, 0.11, 0.189, 0.166, and 0.187, respectively. Thus, choosing $\epsilon_{k_t} = 0.1$ is a sensible pick. Any pick too small (i.e., $\epsilon_{k_t} \rightarrow 0$) would get us right around the median value, which defeats the purpose of establish some bias towards favorable indeterminacy. Any pick too large creates a significant gap between attributes with favorable indeterminacy and those with unfavorable indeterminacy. Ultimately, having the intervals available to the decision maker empowers them in choosing any approximation approach that suits the end-goal of the scenario at hand.

4 Future work and discussion

We have developed a DSS computation model for decision-support scenarios where we have N objectives with m attributes spanning over n time periods. We represented each attribute in the form of a single-valued neutrosophic set and performed necessary operations to determine a specific set of strategies in which each attribute's imprecise consequence were presented as a continuous distribution interval. We proceeded with defining a model to detect when indeterminacy is favorable or unfavorable, and presented approaches that help achieve that bias. We provided a computation example in which the model is used, and conducted a sensitivity analysis of the results. While this example provided a clear application of the model, certain areas still need further discussion. These areas, which can be

addressed in future research on this computation model, include the following:

1. Since each imprecise consequence is given in the form of a continuous distribution interval, assuming that the distribution is normal, would the normal density of each interval help establish a trend about each attribute?
2. Can the neutrosophic components of an attribute be linked with any property other than how impactful the attribute would be in the future?

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6 Appendix

This appendix section contains tables that present the data used and the obtained results for the model computation example. A summary is as follows:

1. Table 1: Initial neutrosophic values for all attributes at time $t = 0$.
2. Table 2: Neutrosophic values for all attributes at $1 \leq t \leq 10$.
3. Table 3: Score (σ), accuracy (α), and certainty (υ) values for all attributes' SVNSs, at time $t \in [0, 10]$.
4. Table 4: Imprecise consequence intervals for all attributes' SVNSs, at time $t \in [0, 10]$.
5. Table 5: Imprecise consequence intervals medians for at time t, for each a_{k_t}.
6. Table 6: Imprecise consequence weight values at time t for each a_{k_t}, obtained via the normalization of the imprecise consequence values of a_{k_t} for each t.

7. Table 7: Imprecise consequence weight values at future time t for each a_{k_t}, obtained via adding or subtracting an arbitrary $\epsilon_{k_t} = 0.1$ from the imprecise consequence interval median for each attribute a_{k_t}, depending on whether the value of i_{k_t} is less than 0.5 or not, at time $t > 0$.

Table 1 Initial neutrosophic values for all attributes, at time $t = 0$.

Neutrosophic Component	Attributes $\left(a_{k_t}(t=0)\right)$									
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
τ_k	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
i_k	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
f_k	0.419	0.26	0.851	0.742	0.03	0.485	0.435	0.856	0.075	0.366

Table 2 Neutrosophic values for all attributes, at $1 \leq t \leq 10$.

Neutrosophic Component	Attributes $\left(a_{k_t}(t>0)\right)$									
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
	$a_{k_t}(t=1)$									
τ_k	0.913	0.636	0.624	0.374	0.504	0.989	0.471	0.009	0.35	0.477
i_k	0.434	0.861	0.749	0.829	0.001	0.29	0.981	0.867	0.755	0.512
f_k	0.087	0.364	0.376	0.626	0.496	0.011	0.529	0.991	0.65	0.523
	$a_{k_t}(t=2)$									
τ_k	0.553	0.77	0.519	0.293	0.246	0.701	0.532	0.276	0.094	0.903
i_k	0.966	0.126	0.951	0.797	0.544	0.373	0.782	0.596	0.482	0.033
f_k	0.447	0.23	0.481	0.707	0.754	0.299	0.468	0.724	0.906	0.097
	$a_{k_t}(t=3)$									
τ_k	0.223	0.256	0.014	0.898	0.026	0.994	0.851	0.887	0.704	0.257
i_k	0.965	0.619	0.537	0.066	0.22	0.111	0.06	0.125	0.252	0.535
f_k	0.777	0.744	0.986	0.102	0.974	0.006	0.149	0.113	0.296	0.743
	$a_{k_t}(t=4)$									
τ_k	0.227	0.505	0.254	0.872	0.116	0.634	0.274	0.614	0.904	0.1
i_k	0.4	0.548	0.012	0.39	0.843	0.129	0.558	0.305	0.198	0.574
f_k	0.773	0.495	0.746	0.128	0.884	0.366	0.726	0.386	0.096	0.9
	$a_{k_t}(t=5)$									
τ_k	0.345	0.103	0.481	0.036	0.01	0.692	0.479	0.85	0.495	0.187
i_k	0.048	0.66	0.083	0.333	0.306	0.065	0.568	0.354	0.349	0.379
f_k	0.655	0.897	0.519	0.964	0.99	0.308	0.521	0.15	0.505	0.813
	$a_{k_t}(t=6)$									
τ_k	0.421	0.236	0.379	0.072	0.582	0.598	0.794	0.837	0.553	0.041
i_k	0.115	0.947	0.764	0.556	0.805	0.948	0.426	0.043	0.408	0.503
f_k	0.579	0.764	0.621	0.928	0.418	0.402	0.206	0.163	0.447	0.959
	$a_{k_t}(t=7)$									
τ_k	0.431	0.498	0.436	0.27	0.235	0.004	0.468	0.334	0.564	0.568
i_k	0.197	0.046	0.041	0.591	0.143	0.081	0.875	0.682	0.014	0.159
f_k	0.569	0.502	0.564	0.73	0.765	0.996	0.532	0.666	0.436	0.432
	$a_{k_t}(t=8)$									
τ_k	0.767	0.571	0.761	0.344	0.032	0.168	0.239	0.807	0.359	0.051
i_k	0.265	0.164	0.436	0.68	0.054	0.778	0.514	0.228	0.855	0.846
f_k	0.233	0.429	0.239	0.656	0.968	0.832	0.761	0.193	0.641	0.949
	$a_{k_t}(t=9)$									
τ_k	0.005	0.318	0.816	0.064	0.286	0.337	0.622	0.457	0.09	0.554
i_k	0.028	0.749	0.314	0.722	0.915	0.475	0.687	0.37	0.297	0.473
f_k	0.995	0.682	0.184	0.936	0.714	0.663	0.378	0.543	0.91	0.446
	$a_{k_t}(t=10)$									
τ_k	0.738	0.788	0.46	0.017	0.401	0.304	0.657	0.921	0.367	0.925
i_k	0.447	0.399	0.973	0.962	0.626	0.191	0.379	0.201	0.957	0.162
f_k	0.262	0.212	0.54	0.983	0.599	0.696	0.343	0.079	0.633	0.075

Table 3 Score (σ), accuracy (α), and certainty (v) values for all attributes' SVNSs, at time $t \in [0, 10]$.

Function	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
					Attributes $\left(a_{k_t}\right)$					
					$a_{k_t}(t=0)$					
$\sigma_{a_{k_t}}(x)$	0.721	0.827	0.433	0.505	0.98	0.677	0.71	0.429	0.95	0.756
$\alpha_{a_{k_t}}(x)$	0.162	0.48	-0.702	-0.484	0.94	0.03	0.13	-0.712	0.85	0.268
$v_{a_{k_t}}(x)$	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
					$a_{k_t}(t=1)$					
$\sigma_{a_{k_t}}(x)$	0.797	0.47	0.5	0.306	0.669	0.896	0.32	0.05	0.315	0.481
$\alpha_{a_{k_t}}(x)$	0.826	0.272	0.248	-0.252	0.008	0.978	-0.058	-0.982	-0.3	-0.046
$v_{a_{k_t}}(x)$	0.913	0.636	0.624	0.374	0.504	0.989	0.471	0.009	0.35	0.477
					$a_{k_t}(t=2)$					
$\sigma_{a_{k_t}}(x)$	0.38	0.805	0.362	0.263	0.316	0.676	0.427	0.319	0.235	0.924
$\alpha_{a_{k_t}}(x)$	0.106	0.54	0.038	-0.414	-0.508	0.402	0.064	-0.448	-0.812	0.806
$v_{a_{k_t}}(x)$	0.553	0.77	0.519	0.293	0.246	0.701	0.532	0.276	0.094	0.903
					$a_{k_t}(t=3)$					
$\sigma_{a_{k_t}}(x)$	0.16	0.298	0.164	0.91	0.277	0.959	0.881	0.883	0.719	0.326
$\alpha_{a_{k_t}}(x)$	-0.554	-0.488	-0.972	0.796	-0.948	0.988	0.702	0.774	0.408	-0.486
$v_{a_{k_t}}(x)$	0.223	0.256	0.014	0.898	0.026	0.994	0.851	0.887	0.704	0.257
					$a_{k_t}(t=4)$					
$\sigma_{a_{k_t}}(x)$	0.351	0.487	0.499	0.785	0.13	0.713	0.33	0.641	0.87	0.209
$\alpha_{a_{k_t}}(x)$	-0.546	0.01	-0.492	0.744	-0.768	0.268	-0.452	0.228	0.808	-0.8
$v_{a_{k_t}}(x)$	0.227	0.505	0.254	0.872	0.116	0.634	0.274	0.614	0.904	0.1
					$a_{k_t}(t=5)$					
$\sigma_{a_{k_t}}(x)$	0.547	0.182	0.626	0.246	0.238	0.773	0.463	0.782	0.547	0.332
$\alpha_{a_{k_t}}(x)$	-0.31	-0.794	-0.038	-0.928	-0.98	0.384	-0.042	0.7	-0.01	-0.626
$v_{a_{k_t}}(x)$	0.345	0.103	0.481	0.036	0.01	0.692	0.479	0.85	0.495	0.187
					$a_{k_t}(t=6)$					
$\sigma_{a_{k_t}}(x)$	0.576	0.175	0.331	0.196	0.453	0.416	0.721	0.877	0.566	0.193
$\alpha_{a_{k_t}}(x)$	-0.158	-0.528	-0.242	-0.856	0.164	0.196	0.588	0.674	0.106	-0.918
$v_{a_{k_t}}(x)$	0.421	0.236	0.379	0.072	0.582	0.598	0.794	0.837	0.553	0.041
					$a_{k_t}(t=7)$					
$\sigma_{a_{k_t}}(x)$	0.555	0.65	0.61	0.316	0.442	0.309	0.354	0.329	0.705	0.659
$\alpha_{a_{k_t}}(x)$	-0.138	-0.004	-0.128	-0.46	-0.53	-0.992	-0.064	-0.332	0.128	0.136
$v_{a_{k_t}}(x)$	0.431	0.498	0.436	0.27	0.235	0.004	0.468	0.334	0.564	0.568
					$a_{k_t}(t=8)$					
$\sigma_{a_{k_t}}(x)$	0.756	0.659	0.695	0.336	0.337	0.186	0.321	0.795	0.288	0.085
$\alpha_{a_{k_t}}(x)$	0.534	0.142	0.522	-0.312	-0.936	-0.664	-0.522	0.614	-0.282	-0.898
$v_{a_{k_t}}(x)$	0.767	0.571	0.761	0.344	0.032	0.168	0.239	0.807	0.359	0.051
					$a_{k_t}(t=9)$					
$\sigma_{a_{k_t}}(x)$	0.327	0.296	0.773	0.135	0.219	0.4	0.519	0.515	0.294	0.545
$\alpha_{a_{k_t}}(x)$	-0.99	-0.364	0.632	-0.872	-0.428	-0.326	0.244	-0.086	-0.82	0.108
$v_{a_{k_t}}(x)$	0.005	0.318	0.816	0.064	0.286	0.337	0.622	0.457	0.09	0.554
					$a_{k_t}(t=10)$					
$\sigma_{a_{k_t}}(x)$	0.676	0.726	0.316	0.024	0.392	0.472	0.645	0.88	0.259	0.896
$\alpha_{a_{k_t}}(x)$	0.476	0.576	-0.08	-0.966	-0.198	-0.392	0.314	0.842	-0.266	0.85
$v_{a_{k_t}}(x)$	0.738	0.788	0.46	0.017	0.401	0.304	0.657	0.921	0.367	0.925

Table 4 Imprecise consequence intervals for all attributes' SVNSs, at time $t \in [0, 10]$.

Imprecise Consequences Set (S_q) of Strategies Set S				
s_{q_1}	s_{q_2}	s_{q_3}	s_{q_4}	s_{q_5}
s_{q_6}	s_{q_7}	s_{q_8}	s_{q_9}	$s_{q_{10}}$
$t = 0$				
0.581	0.74	0.149	0.258	0.97
0.515	0.565	0.144	0.925	0.634
$t = 1$				
[0.753, 0.797]	[0.47, 0.711]	[0.5, 0.666]	[0.306, 0.526]	[0.336, 0.669]
[0.756, 0.896]	[0.32, 0.641]	[0.05, 0.295]	[0.315, 0.485]	[0.481, 0.489]
$t = 2$				
[0.38,0.691]	[0.555,0.805]	[0.362,0.663]	[0.263,0.461]	[0.316,0.345]
[0.592,0.676]	[0.427,0.615]	[0.319,0.383]	[0.223,0.235]	[0.613,0.924]
$t = 3$				
[0.16,0.47]	[0.298,0.377]	[0.164,0.188]	[0.621,0.91]	[0.091,0.277]
[0.7,0.959]	[0.587,0.881]	[0.633,0.883]	[0.553,0.719]	[0.326,0.35]
$t = 4$				
[0.285,0.351]	[0.487,0.519]	[0.173,0.499]	[0.711,0.785]	[0.13,0.358]
[0.466,0.713]	[0.33,0.369]	[0.511,0.641]	[0.669,0.87]	[0.209,0.258]
$t = 5$				
[0.246,0.547]	[0.182,0.289]	[0.348,0.626]	[0.135,0.246]	[0.109,0.238]
[0.483,0.773]	[0.463,0.509]	[0.685,0.782]	[0.446,0.547]	[0.251,0.332]
$t = 6$				
[0.319,0.576]	[0.175,0.473]	[0.331,0.507]	[0.196,0.233]	[0.453,0.656]
[0.416,0.715]	[0.671,0.721]	[0.572,0.877]	[0.505,0.566]	[0.193,0.195]
$t = 7$				
[0.353,0.555]	[0.347,0.65]	[0.304,0.61]	[0.316,0.377]	[0.204,0.442]
[0.03,0.309]	[0.354,0.604]	[0.329,0.45]	[0.381,0.705]	[0.432,0.659]
$t = 8$				
[0.6,0.756]	[0.435,0.659]	[0.653,0.695]	[0.336,0.456]	[0.039,0.337]
[0.186,0.371]	[0.321,0.331]	[0.614,0.795]	[0.288,0.524]	[0.085,0.316]
$t = 9$				
[0.013,0.327]	[0.296,0.462]	[0.649,0.773]	[0.135,0.283]	[0.219,0.496]
[0.383,0.4]	[0.519,0.644]	[0.428,0.515]	[0.159,0.294]	[0.527,0.545]
$t = 10$				
[0.641,0.676]	[0.658,0.726]	[0.316,0.631]	[0.024,0.332]	[0.392,0.476]
[0.266,0.472]	[0.564,0.645]	[0.681,0.88]	[0.259,0.564]	[0.671,0.896]

Table 5 Imprecise consequence intervals medians for at time t, for each a_{k_t}.

Time (t)	Imprecise Consequence Median Values $(s_{\bar{q}_k})$ for a_{k_t} at Time t									
	$s_{\bar{q}_1}$	$s_{\bar{q}_2}$	$s_{\bar{q}_3}$	$s_{\bar{q}_4}$	$s_{\bar{q}_5}$	$s_{\bar{q}_6}$	$s_{\bar{q}_7}$	$s_{\bar{q}_8}$	$s_{\bar{q}_9}$	$s_{\bar{q}_{10}}$
$t = 0$	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
$t = 1$	0.775	0.59	0.583	0.416	0.503	0.826	0.481	0.172	0.4	0.485
$t = 2$	0.536	0.68	0.512	0.362	0.33	0.634	0.521	0.351	0.229	0.768
$t = 3$	0.315	0.338	0.176	0.766	0.184	0.829	0.734	0.758	0.636	0.338
$t = 4$	0.318	0.503	0.336	0.748	0.244	0.59	0.35	0.576	0.77	0.233
$t = 5$	0.396	0.236	0.487	0.19	0.174	0.628	0.486	0.734	0.497	0.292
$t = 6$	0.448	0.324	0.419	0.215	0.554	0.566	0.696	0.724	0.536	0.194
$t = 7$	0.454	0.498	0.457	0.347	0.323	0.169	0.479	0.39	0.543	0.546
$t = 8$	0.678	0.547	0.674	0.396	0.188	0.278	0.326	0.704	0.406	0.2
$t = 9$	0.17	0.379	0.711	0.209	0.358	0.392	0.582	0.472	0.226	0.536
$t = 10$	0.659	0.692	0.474	0.178	0.434	0.369	0.604	0.78	0.412	0.784

Table 6 Imprecise consequence weight values at time t for each a_{k_t}, obtained via the normalization of the imprecise consequence values of a_{k_t} for each t.

Time (t)	Imprecise Consequence Weighted Values $(s_{\hat{q}_k})$ for a_{k_t} at Time t										$\sum_{k=1}^{m} s_{\hat{q}_k}$
	$s_{\hat{q}_1}$	$s_{\hat{q}_2}$	$s_{\hat{q}_3}$	$s_{\hat{q}_4}$	$s_{\hat{q}_5}$	$s_{\hat{q}_6}$	$s_{\hat{q}_7}$	$s_{\hat{q}_8}$	$s_{\hat{q}_9}$	$s_{\hat{q}_{10}}$	
$t = 0$	0.106	0.135	0.027	0.047	0.177	0.094	0.103	0.026	0.169	0.116	1.00
$t = 1$	0.148	0.113	0.111	0.08	0.096	0.158	0.092	0.033	0.076	0.093	1.00
$t = 2$	0.109	0.138	0.104	0.074	0.067	0.129	0.106	0.071	0.047	0.156	1.00
$t = 3$	0.062	0.067	0.035	0.151	0.036	0.163	0.145	0.149	0.125	0.067	1.00
$t = 4$	0.068	0.108	0.072	0.16	0.052	0.126	0.075	0.123	0.165	0.05	1.00
$t = 5$	0.096	0.057	0.118	0.046	0.042	0.152	0.118	0.178	0.121	0.071	1.00
$t = 6$	0.096	0.069	0.09	0.046	0.118	0.121	0.149	0.155	0.115	0.041	1.00
$t = 7$	0.108	0.118	0.109	0.083	0.077	0.04	0.114	0.093	0.129	0.13	1.00
$t = 8$	0.154	0.124	0.153	0.09	0.043	0.063	0.074	0.16	0.092	0.045	1.00
$t = 9$	0.042	0.094	0.176	0.052	0.089	0.097	0.144	0.117	0.056	0.133	1.00
$t = 10$	0.122	0.128	0.088	0.033	0.081	0.069	0.112	0.145	0.076	0.146	1.00

Table 7 Imprecise consequence weight values at future time t for each a_{k_t}, obtained via adding or subtracting an arbitrary $\epsilon_{k_t} = 0.1$ from the imprecise consequence interval median for each attribute a_{k_t}, depending on whether the value of i_{k_t} is less than 0.5 or not, at time $t > 0$.

Time ($t > 0$)	Imprecise Consequence Weighted Values ($s_{\hat{q}_k}$) for a_{k_t} at Future t										$\sum_{k=1}^{m} s_{\hat{q}_k}$
	$s_{\hat{q}_1}$	$s_{\hat{q}_2}$	$s_{\hat{q}_3}$	$s_{\hat{q}_4}$	$s_{\hat{q}_5}$	$s_{\hat{q}_6}$	$s_{\hat{q}_7}$	$s_{\hat{q}_8}$	$s_{\hat{q}_9}$	$s_{\hat{q}_{10}}$	
$t = 1$	0.167	0.094	0.092	0.06	0.115	0.177	0.073	0.014	0.057	0.074	0.923
$t = 2$	0.089	0.158	0.084	0.053	0.047	0.149	0.086	0.051	0.067	0.176	0.96
$t = 3$	0.042	0.047	0.015	0.171	0.056	0.183	0.164	0.169	0.145	0.047	1.039
$t = 4$	0.09	0.086	0.093	0.182	0.031	0.148	0.054	0.145	0.186	0.028	1.043
$t = 5$	0.12	0.033	0.142	0.07	0.067	0.177	0.094	0.202	0.145	0.095	1.145
$t = 6$	0.117	0.048	0.068	0.025	0.097	0.1	0.17	0.176	0.136	0.02	0.957
$t = 7$	0.132	0.142	0.132	0.059	0.101	0.064	0.09	0.069	0.153	0.154	1.096
$t = 8$	0.177	0.147	0.176	0.067	0.065	0.04	0.051	0.183	0.07	0.023	0.999
$t = 9$	0.067	0.069	0.201	0.027	0.064	0.122	0.119	0.142	0.081	0.158	1.05
$t = 10$	0.141	0.147	0.069	0.014	0.062	0.087	0.131	0.163	0.058	0.164	1.036