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An approach to neutrosophic subgroup and its fundamental properties

Vildan Çetkin* and Halis Aygün
Department of Mathematics, Kocaeli University, Umuttepe Campus, Kocaeli, Turkey

Abstract. In this paper, we examine the group structure of single valued neutrosophic sets. We introduce an approach to neutrosophic subgroup and establish some of its basic properties and characterizations. Then we give the homomorphic image and preimage of a neutrosophic (normal) subgroup.

Keywords: Neutrosophic set, single valued neutrosophic set, group, normal group, homomorphism

1. Introduction

In many practical situations and in many complex systems like biological, behavioral and chemical etc., different types of uncertainties are encountered. The concept of fuzzy set was introduced by Zadeh [14] to handle uncertainties in many real applications. The traditional fuzzy set is characterized by the membership value or the grade of membership value. Sometimes it may be difficult to assign the membership value for a fuzzy set. Consequently the concept of interval valued fuzzy set [11] was proposed to capture the uncertainty of grade of membership value. In some real life problems in expert system, belief system, information fusion and so on, we must consider the truth-membership as well as the falsity-membership for proper description of an object in uncertain, ambiguous environment. Intuitionistic fuzzy set introduced by Atanassov [2] is appropriate for such a situation. Neutrosophy was introduced by Smarandache [10] in 1999 to handle the indeterminate information and inconsistent information which exist commonly in real situations. "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra" [10]. In the neutrosophic set, a truth-membership, an indeterminacy-membership, and a falsity-membership are represented independently. Neutrosophic set generalizes the above mentioned sets from philosophical point of view. From scientific and engineering point of view, the definition of neutrosophic set was specified by Wang et al. [12] which is called single valued neutrosophic set. The single valued neutrosophic set is a generalization of classical set, fuzzy set, intuitionistic fuzzy set and paraconsistent set etc. Neutrosophic set is applied to algebraic and topological directions (see [1, 3–5, 8, 9]). Rosenfeld [7] initiated the concept of fuzzy subgroups in 1971 and then so many contributions were made on these main direction. In [13], Xiaoping studied intuitionistic fuzzy normal subgroup. Palaniappan et al. [6] gave the definition of intuitionistic L-fuzzy subgroup and studied some of its properties. In this work, we give an approach to group structure of single valued neutrosophic sets. We define neutrosophic normal subgroup and give some properties of these structures. Moreover, we define image and preimage of a (single valued) neutrosophic set and examine homomorphic image and preimage of a neutrosophic (normal) subgroup. By this way, we obtain the generalized form of the fuzzy subgroup and intuitionistic fuzzy subgroup of a classical group.

*Corresponding author. Vildan Çetkin, Department of Mathematics, Kocaeli University, Umuttepe Campus, 41380, Kocaeli, Turkey. Tel.: +90 262 303 2122; Fax: +90 262 303 2003; E-mail: vcetkin@gmail.com.

2. Preliminaries

In this chapter, we give some preliminaries about single valued neutrosophic sets and set operations, which will be called neutrosophic sets, for simplicity.

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Definition 2.1. [10] A neutrosophic set A on the universe of X is defined as

$A = \{< x, t_A(x), i_A(x), f_A(x) >, x \in X\}$

where $t_A, i_A, f_A : X \to]^-0, 1^+[$ and

$^-0 \leq t_A(x) + i_A(x) + f_A(x) \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non standard subsets of $]^-0, 1^+[$. But in real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence throughout this work, the following specified definition of a neutrosophic set known as single valued neutrosophic set is considered.

Definition 2.2. [12] Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set (SVNS) A on X is characterized by truth-membership function t_A, indeterminacy-membership function i_A and falsity-membership function f_A. For each point x in X, $t_A(x)$, $i_A(x)$, $f_A(x) \in [0, 1]$.

A neutrosophic set A can be written as

$$A = \sum_{i=1}^{n} < t(x_i), i(x_i), f(x_i) > /x_i, \ x_i \in X.$$

Example 2.3.[12] Assume that $X = \{x_1, x_2, x_3\}$, x_1 is capability, x_2 is trustworthiness and x_3 is price. The values of x_1, x_2 and x_3 are in [0, 1]. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". A is a single valued neutrosophic set of X defined by

$A =< 0.3, 0.4, 0.5 > /x_1+ < 0.5, 0.2, 0.3 > /x_2+ < 0.7, 0.2, 0.2 > /x_3$.

Since the membership functions t_A, i_A, f_A are defined from X into the unit interval [0, 1] as $t_A, i_A, f_A : X \to [0, 1]$, a (single valued) neutrosophic set A will be denoted by a mapping defined as $A : X \to [0, 1] \times [0, 1] \times [0, 1]$ and $A(x) = (t_A(x), i_A(x), f_A(x))$, for simplicity.

Definition 2.4. [8, 12] Let A and B be two neutrosophic sets on X. Then

(1) A is contained in B, denoted as $A \subseteq B$, if and only if $A(x) \leq B(x)$. This means that $t_A(x) \leq t_B(x)$, $i_A(x) \leq i_B(x)$ and $f_A(x) \geq f_B(x)$. Two sets A and B is called equal, i.e., $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(2) the union of A and B is denoted by $C = A \cup B$ and defined as $C(x) = A(x) \vee B(x)$ where $A(x) \vee B(x) = (t_A(x) \vee t_B(x), i_A(x) \vee i_B(x), f_A(x) \wedge f_B(x))$, for each $x \in X$. This means that

$$t_C(x) = \max\{t_A(x), t_B(x)\},$$
$$i_C(x) = \max\{i_A(x), i_B(x)\} \text{ and}$$
$$f_C(x) = \min\{f_A(x), f_B(x)\}.$$

(3) the intersection of A and B is denoted by $C = A \cap B$ and defined as $C(x) = A(x) \wedge B(x)$ where $A(x) \wedge B(x) = (t_A(x) \wedge t_B(x), i_A(x) \wedge i_B(x), f_A(x) \vee f_B(x))$, for each $x \in X$. This means that

$$t_C(x) = \min\{t_A(x), t_B(x)\},$$
$$i_C(x) = \min\{i_A(x), i_B(x)\} \text{ and}$$
$$f_C(x) = \max\{f_A(x), f_B(x)\}.$$

(4) the complement of A is denoted by A^c and defined as $A^c(x) = (f_A(x), 1 - i_A(x), t_A(x))$, for each $x \in X$. Here $(A^c)^c = A$.

Proposition 2.5. [12] Let A, B and C be the neutrosophic sets on the common universe X. Then the following properties are valid.

(1) $A \cup B = B \cup A$, $A \cap B = B \cap A$.
(2) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.
(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(4) $A \cap \emptyset = \emptyset$, $A \cup \emptyset = A$, $A \cup X = X$, $A \cap X = A$, where $t_\emptyset = i_\emptyset = 0$, $f_\emptyset = 1$ and $t_X = i_X = 1$, $f_X = 0$.
(5) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Definition 2.6. Let $g : X_1 \to X_2$ be a function and A, B be the neutrosophic sets of X_1 and X_2, respectively. Then the image of a neutrosophic set A is a neutrosophic set of X_2 and it is defined as follows:

$$\begin{aligned} g(A)(y) &= (t_{g(A)}(y), i_{g(A)}(y), f_{g(A)}(y)) \\ &= (g(t_A)(y), g(i_A)(y), g(f_A)(y)), \end{aligned}$$

for all $y \in X_2$ where

$$g(t_A)(y) = \begin{cases} \bigvee t_A(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases},$$

$$g(i_A)(y) = \begin{cases} \bigvee i_A(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases},$$

$$g(f_A)(y) = \begin{cases} \bigwedge f_A(x), & \text{if } x \in g^{-1}(y); \\ 1, & \text{otherwise.} \end{cases}$$

And the preimage of a neutrosophic set B is a neutrosophic set of X_1 and it is defined as follows: for all $x \in X_1$

$$\begin{aligned} g^{-1}(B)(x) &= (t_{g^{-1}(B)}(x), i_{g^{-1}(B)}(x), f_{g^{-1}(B)}(x)) \\ &= (t_B(g(x)), i_B(g(x)), f_B(g(x))) \\ &= B(g(x)) \end{aligned}$$

3. Neutrosophic subgroup

Definition 3.1. Let $(X, \cdot)$ be a classical group and A be a neutrosophic set on X. A is called a neutrosophic subgroup of X if the following conditions are satisfied: for each $x, y \in X$,

(N1)$A(x \cdot y) \geq A(x) \wedge A(y)$, i.e.,

$t_A(x \cdot y) \geq t_A(x) \wedge t_A(y)$, $i_A(x \cdot y) \geq i_A(x) \wedge i_A(y)$ and $f_A(x \cdot y) \leq f_A(x) \vee f_A(y)$.

(N2) $A(x^{-1}) \geq A(x)$, i.e.,

$t_A(x^{-1}) \geq t_A(x)$, $i_A(x^{-1}) \geq i_A(x)$ and $f_A(x^{-1}) \leq f_A(x)$.

The collection of all neutrosophic subgroups of X is denoted by $NS(X)$.

Example 3.2. Let us take into consideration the classical group $X = \{1, -1, i, -i\}$ with the natural multiplication. Define the neutosophic set A on X as follows:

$$\begin{aligned} A = \{&< 0.6, 0.3, 0.5 > /1 + < 0.7, 0.4, 0.3 > / \\ &-1 + < 0.8, 0.4, 0.2 > /i + < 0.8, 0.4, 0.2 > / - i\}. \end{aligned}$$

It is clear that the neutrosophic set A is a neutrosophic subgroup of X.

Theorem 3.3. *Let X be a classical group and A be a neutrosophic subgroup of X. Then the following properties are satisfied:*

(1) $A(e) \geq A(x)$, for all $x \in X$, where e is the unit element of X.
(2) $A(x^{-1}) = A(x)$, for each $x \in X$.

Proof. (1) Let e be the unit element of X and $x \in X$ be arbitrary, then by (N1), (N2) of Definition 3.1,

$$\begin{aligned} t_A(e) &= t_A(x \cdot x^{-1}) \\ &\geq t_A(x) \wedge t_A(x^{-1}) \\ &\geq t_A(x) \wedge t_A(x) = t_A(x). \end{aligned}$$

$$\begin{aligned} f_A(e) &= f_A(x \cdot x^{-1}) \\ &\leq f_A(x) \vee f_A(x^{-1}) \\ &\leq f_A(x) \vee f_A(x) = f_A(x). \end{aligned}$$

From the similar idea, it is easily shown that $i_A(e) \geq i_A(x)$. Hence, the desired inequality $A(e) \geq A(x)$ is satisfied, for all $x \in X$.

(2) Let $x \in X$ be given. Since A is a neutrosophic subgroup of X, $A(x^{-1}) \geq A(x)$ is clear from (N2). Again by applying (N2) and using group structure of X, the other side of the inequality is proved as follows:

$$\begin{aligned} t_A(x) &= t_A((x^{-1})^{-1}) \geq t_A(x^{-1}), \\ i_A(x) &= i_A((x^{-1})^{-1}) \geq i_A(x^{-1}) \text{ and} \\ f_A(x) &= f_A((x^{-1})^{-1}) \leq f_A(x^{-1}). \text{ So,} \\ A(x^{-1}) &= (t_A(x^{-1}), i_A(x^{-1}), f_A(x^{-1})) \\ &= (t_A(x), i_A(x), f_A(x)) = A(x). \end{aligned}$$

Theorem 3.4. *Let X be a classical group and A be a neutrosophic set on X. Then $A \in NS(X)$ if and only if $A(x \cdot y^{-1}) \geq A(x) \wedge A(y)$, for each $x, y \in X$.*

Proof. Let A be a neutrosophic subgroup of X and $x, y \in X$. So, it is clear that

$$t_A(x \cdot y^{-1}) \geq t_A(x) \wedge t_A(y^{-1}) = t_A(x) \wedge t_A(y).$$

According to the similar discussion, the following inequalities are also true:

$$\begin{aligned} &i_A(x \cdot y^{-1}) \geq i_A(x) \wedge i_A(y) \text{ and} \\ &f_A(x \cdot y^{-1}) \leq f_A(x) \vee f_A(y). \text{ Hence,} \\ &A(x \cdot y^{-1}) = (t_A(x \cdot y^{-1}), i_A(x \cdot y^{-1}), f_A(x \cdot y^{-1})) \\ &\geq (t_A(x) \wedge t_A(y), i_A(x) \wedge i_A(y), f_A(x) \vee f_A(y)) \\ &= (t_A(x), i_A(x), f_A(x)) \wedge (t_A(y), i_A(y), f_A(y)) \\ &= A(x) \wedge A(y). \end{aligned}$$

Conversely, let e be the unit of X. Since X is a classical group,

$$\begin{aligned} t_A(x^{-1}) &= t_A(e \cdot x^{-1}) \\ &\geq t_A(e) \wedge t_A(x) \\ &= t_A(x \cdot x^{-1}) \wedge t_A(x) \\ &\geq t_A(x) \wedge t_A(x) \wedge t_A(x) = t_A(x). \end{aligned}$$

Similarly, $i_A(x^{-1}) \geq i_A(x)$ and $f_A(x^{-1}) \leq f_A(x)$. So, the condition (N2) of Definition 3.1 is satisfied. Now let us show the condition (N1),

$$f_A(x \cdot y) = f_A(x \cdot (y^{-1})^{-1}) \leq f_A(x) \vee f_A(y^{-1}) \leq f_A(x) \vee f_A(y).$$

Also, the inequalities $t_A(x \cdot y) \geq t_A(x) \wedge t_A(y)$ and $i_A(x \cdot y) \geq i_A(x) \wedge i_A(y)$ are clear. Therefore, (N1) of Definition 3.1 is also satisfied.

Theorem 3.5. *Let X be a classical group and A, B be two neutrosophic sets on X. If A, B are neutrosophic subgroups of X, then the intersection $A \cap B$ so is.*

Proof. Let $x, y \in X$ be arbitrary. By Theorem 3.4, it is enough to show that

$$(A \cap B)(x \cdot y^{-1}) \geq (A \cap B)(x) \wedge (A \cap B)(y), \text{ i.e.,}$$

$$t_{A\cap B}(x \cdot y^{-1}) \geq t_{A\cap B}(x) \wedge t_{A\cap B}(y),$$

$$i_{A\cap B}(x \cdot y^{-1}) \geq i_{A\cap B}(x) \wedge i_{A\cap B}(y)$$

$$\text{and } f_{A\cap B}(x \cdot y^{-1}) \leq f_{A\cap B}(x) \vee f_{A\cap B}(y).$$

First consider the truth-membership degree of the intersection,

$$\begin{aligned} t_{A\cap B}(x \cdot y^{-1}) &= t_A(x \cdot y^{-1}) \wedge t_B(x \cdot y^{-1}) \\ &\geq (t_A(x) \wedge t_A(y)) \wedge (t_B(x) \wedge t_B(y)) \\ &= (t_A(x) \wedge t_B(x)) \wedge (t_A(y) \wedge t_B(y)) \\ &= t_{A\cap B}(x) \wedge t_{A\cap B}(y). \end{aligned}$$

The other inequalities are similarly proved. Therefore, $A \cap B \in NS(X)$.

Let A be a neutrosophic set on X and $\alpha \in [0, 1]$. Define the α-level sets of A as follows:

$$(t_A)_\alpha = \{x \in X \mid t_A(x) \geq \alpha\},$$

$$(i_A)_\alpha = \{x \in X \mid i_A(x) \geq \alpha\}, \text{ and}$$

$$(f_A)^\alpha = \{x \in X \mid f_A(x) \leq \alpha\}.$$

It is easy to verify that

(1) If $A \subseteq B$ and $\alpha \in [0, 1]$, then

$$(t_A)_\alpha \subseteq (t_B)_\alpha, (i_A)_\alpha \subseteq (i_B)_\alpha, \text{ and } (f_A)^\alpha \supseteq (f_B)^\alpha.$$

(2) $\alpha \leq \beta$ implies $(t_A)_\alpha \supseteq (t_A)_\beta$, $(i_A)_\alpha \supseteq (i_A)_\beta$, and $(f_A)^\alpha \subseteq (f_A)^\beta$.

Proposition 3.6. *A is a neutrosophic subgroup of a classical group X if and only if for all $\alpha \in [0, 1]$, α-level sets of A, $(t_A)_\alpha, (i_A)_\alpha$ and $(f_A)^\alpha$ are classical subgroups of X.*

Proof. Let A be a neutrosophic subgroup of X, $\alpha \in [0, 1]$ and $x, y \in (t_A)_\alpha$(similarly $x, y \in (i_A)_\alpha, (f_A)^\alpha$). By the assumption,

$t_A(x \cdot y^{-1}) \geq t_A(x) \wedge t_A(y) \geq \alpha \wedge \alpha = \alpha$ (and similarly, $i_A(x \cdot y^{-1}) \geq \alpha$ and $f_A(x \cdot y^{-1}) \leq \alpha$). Hence $x \cdot y^{-1} \in (t_A)_\alpha$, (and similarly $x \cdot y^{-1} \in (i_A)_\alpha, (f_A)^\alpha$) for each $\alpha \in [0, 1]$. This means that $(t_A)_\alpha$ (and similarly $(i_A)_\alpha, (f_A)^\alpha$) is a classical subgroup of X for each $\alpha \in [0, 1]$.

Conversely, let $(t_A)_\alpha$ be a classical subgroup of X, for each $\alpha \in [0, 1]$. Let $x, y \in X, \alpha = t_A(x) \wedge t_A(y)$ and $\beta = t_A(x)$. Since $(t_A)_\alpha$ and $(t_A)_\beta$ are classical subgroups of X, $x \cdot y \in (t_A)_\alpha$ and $x^{-1} \in (t_A)_\beta$. Thus, $t_A(x \cdot y) \geq \alpha = t_A(x) \wedge t_A(y)$ and $t_A(x^{-1}) \geq \beta = t_A(x)$. Similarly, $i_A(x \cdot y^{-1}) \geq i_A(x) \wedge i_A(y)$ and $f_A(x \cdot y^{-1}) \leq f_A(x) \vee f_A(y)$. Hence the conditions of Definition 3.1 are satisfied.

Theorem 3.7. *Let X_1, X_2 be the classical groups and $g : X_1 \to X_2$ be a group homomorphism. If A is a neutrosophic subgroup of X_1, then the image of A, $g(A)$ is a neutrosophic subgroup of X_2.*

Proof. Let $A \in NS(X_1)$ and $y_1, y_2 \in X_2$. If $g^{-1}(y_1) = \emptyset$ or $g^{-1}(y_2) = \emptyset$, then it is obvious that $g(A) \in NS(X_2)$. Let us assume that there exist $x_1, x_2 \in X_1$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$. Since g is a group homomorphism,

$$g(t_A)(y_1 \cdot y_2^{-1}) = \bigvee_{y_1 \cdot y_2^{-1} = g(x)} t_A(x) \geq t_A(x_1 \cdot x_2^{-1}),$$

$$g(i_A)(y_1 \cdot y_2^{-1}) = \bigvee_{y_1 \cdot y_2^{-1} = g(x)} i_A(x) \geq i_A(x_1 \cdot x_2^{-1}),$$

$$g(f_A)(y_1 \cdot y_2^{-1}) = \bigwedge_{y_1 \cdot y_2^{-1} = g(x)} f_A(x) \leq f_A(x_1 \cdot x_2^{-1}).$$

By using the above inequalities let us prove that $g(A)(y_1 \cdot y_2^{-1}) \geq g(A)(y_1) \wedge g(A)(y_2)$.

$$\begin{aligned} & g(A)(y_1 \cdot y_2^{-1}) \\ &= (g(t_A)(y_1 \cdot y_2^{-1}), g(i_A)(y_1 \cdot y_2^{-1}), \\ &\quad g(f_A)(y_1 \cdot y_2^{-1})) \\ &= \Bigg(\bigvee_{y_1 \cdot y_2^{-1} = g(x)} t_A(x), \bigvee_{y_1 \cdot y_2^{-1} = g(x)} i_A(x), \\ &\quad \bigwedge_{y_1 \cdot y_2^{-1} = g(x)} f_A(x) \Bigg) \end{aligned}$$

$$\geq (t_A(x_1 \cdot x_2^{-1}), i_A(x_1 \cdot x_2^{-1}), f_A(x_1 \cdot x_2^{-1}))$$
$$\geq (t_A(x_1) \wedge t_A(x_2), i_A(x_1) \wedge i_A(x_2), f_A(x_1) \vee f_A(x_2))$$
$$= (t_A(x_1), i_A(x_1), f_A(x_1)) \wedge (t_A(x_2), i_A(x_2), f_A(x_2)).$$

This is satisfied for each $x_1, x_2 \in X_1$ with $g(x_1) = y_1$ and $g(x_2) = y_2$, then it is obvious that

$$g(A)(y_1 \cdot y_2^{-1})$$
$$\geq \Big(\bigvee_{y_1=g(x_1)} t_A(x_1), \bigvee_{y_1=g(x_1)} i_A(x_1), \bigwedge_{y_1=g(x_1)} f_A(x_1) \Big)$$
$$\wedge \Big(\bigvee_{y_2=g(x_2)} t_A(x_2), \bigvee_{y_2=g(x_2)} i_A(x_2), \bigwedge_{y_2=g(x_2)} f_A(x_2) \Big)$$
$$= (g(t_A)(y_1), g(i_A)(y_1), g(f_A)(y_1))$$
$$\wedge (g(t_A)(y_2), g(i_A)(y_2), g(f_A)(y_2))$$
$$= g(A)(y_1) \wedge g(A)(y_2).$$

Hence the image of a neutrosophic subgroup is also a neutrosophic subgroup.

Theorem 3.8. *Let X_1, X_2 be the classical groups and $g : X_1 \to X_2$ be a group homomorphism. If B is a neutrosophic subgroup of X_2, then the preimage $g^{-1}(B)$ is a neutrosophic subgroup of X_1.*

Proof. Let $B \in NS(X_2)$ and $x_1, x_2 \in X_1$. Since g is a group homomorphism, the following inequality is obtained.

$$g^{-1}(B)(x_1 \cdot x_2^{-1})$$
$$= (t_B(g(x_1 \cdot x_2^{-1})), i_B(g(x_1 \cdot x_2^{-1})), f_B(g(x_1 \cdot x_2^{-1})))$$
$$= (t_B(g(x_1) \cdot g(x_2)^{-1}), i_B(g(x_1) \cdot g(x_2)^{-1}), f_B(g(x_1) \cdot g(x_2)^{-1}))$$
$$\geq (t_B(g(x_1)) \wedge t_B(g(x_2)), i_B(g(x_1)) \wedge i_B(g(x_2)), f_B(g(x_1)) \vee f_B(g(x_2)))$$
$$= (t_B(g(x_1)), i_B(g(x_1)), f_B(g(x_1)))$$
$$\wedge (t_B(g(x_2)), i_B(g(x_2)), f_B(g(x_2)))$$
$$= g^{-1}(B)(x_1) \wedge g^{-1}(B)(x_2).$$

Therefore, $g^{-1}(B) \in NS(X_1)$.

Theorem 3.9. *Let $g : X_1 \to X_2$ be a homomorphism of groups, $A \in NS(X_1)$ and define $A^{-1} : X_1 \to [0, 1] \times [0, 1] \times [0, 1]$ as $A^{-1}(x) = A(x^{-1})$ for arbitrary $x \in X_1$. Then the following properties are valid.*
(1) $A^{-1} \in NS(X_1)$.
(2) $(g(A))^{-1} = g(A^{-1})$.

Proof. It is obvious by the definitions.

Corollary 3.10. *Let $g : X_1 \to X_2$ be an isomorphism of groups, $A \in NS(X_1)$, then $g^{-1}(g(A)) = A$.*

Corollary 3.11. *Let $g : X \to X$ be an isomorphism on a classical group X, $A \in NS(X)$, then $g(A) = A$ if and only if $g^{-1}(A) = A$.*

4. Neutrosophic normal subgroup

Definition 4.1. Let X be a classical group and A be a neutrosophic subgroup of X, then A is called a neutrosophic normal subgroup of X, if $A(x \cdot y \cdot x^{-1}) \geq A(y)$ for all $x, y \in X$. This means that

$$t_A(x \cdot y \cdot x^{-1}) \geq t_A(y),\ i_A(x \cdot y \cdot x^{-1}) \geq i_A(y)$$
$$\text{and } f_A(x \cdot y \cdot x^{-1}) \leq f_A(y), \text{ for all } x, y \in X.$$

The collection all of the neutrosophic normal subgroups of X is denoted by $NNS(X)$.

Theorem 4.2. *Let X be a classical group and $A, B \in NNS(X)$, then $A \cap B \in NNS(X)$.*

Proof. Since A, B are neutrosophic normal subgroups of X, then

$$t_A(x \cdot y \cdot x^{-1}) \geq t_A(y) \text{ and } t_B(x \cdot y \cdot x^{-1}) \geq t_B(y).$$

So by the definition of the intersection,

$$t_{A\cap B}(x \cdot y \cdot x^{-1}) = t_A(x \cdot y \cdot x^{-1})$$
$$\wedge t_B(x \cdot y \cdot x^{-1})$$
$$\geq t_A(y) \wedge t_B(y)$$
$$= t_{A\cap B}(y).$$

By the similar observation,

$$i_{A\cap B}(x \cdot y \cdot x^{-1}) \geq i_{A\cap B}(y) \text{ and}$$
$$f_{A\cap B}(x \cdot y \cdot x^{-1}) \leq f_{A\cap B}(y)$$

are satisfied. Therefore, the intersection of two neutrosophic normal subgroup is also a neutrosophic normal subgroup.

Proposition 4.3. *Let X be a classical group and A be a neutrosophic subgroup of X. Then the followings are equivalent.*

(1) $A \in NNS(X)$.
(2) $A(x \cdot y \cdot x^{-1}) = A(y)$, for all $x, y \in X$.
(3) $A(x \cdot y) = A(y \cdot x)$, for all $x, y \in X$.

Proof. (1) $\Rightarrow$ (2): Let A be a neutrosophic normal subgroup of X. Take $x, y \in X$, then by Definition 4.1,

$$t_A(x \cdot y \cdot x^{-1}) \geq t_A(y),\ i_A(x \cdot y \cdot x^{-1}) \geq i_A(y)$$
$$\text{and } f_A(x \cdot y \cdot x^{-1}) \leq f_A(y).$$

Thus, taking the advantage of the arbitrary property of x, the following is got for the falsity-membership of A,

$$f_A(x^{-1} \cdot y \cdot x) = f_A(x^{-1} \cdot y \cdot (x^{-1})^{-1}) \leq f_A(y).$$

Therefore,

$$\begin{aligned} f_A(y) &= f_A(x^{-1} \cdot (x \cdot y \cdot x^{-1}) \cdot x) \\ &\leq f_A(x \cdot y \cdot x^{-1}) \\ &\leq f_A(y), \end{aligned}$$

i.e., $f_A(x \cdot y \cdot x^{-1}) = f_A(y)$.

Similarly, it is proved that $i_A(x \cdot y \cdot x^{-1}) = i_A(y)$ and $t_A(x \cdot y \cdot x^{-1}) = t_A(y)$.

(2) $\Rightarrow$ (3): Substituting y for $y \cdot x$ in (2), the condition (3) is shown easily.

(3) $\Rightarrow$ (1): According to $A(y \cdot x) = A(x \cdot y)$, the equality $A(x \cdot y \cdot x^{-1}) = A(y \cdot x \cdot x^{-1}) = A(y) \geq A(y)$ is satisfied.

Theorem 4.4. *Let X be a classical group and $A \in NS(X)$. Then $A \in NNS(X)$ if and only if for arbitrary $\alpha \in [0, 1]$, if α-level sets of A are nonempty, then $(t_A)_\alpha, (i_A)_\alpha$ and $(f_A)^\alpha$ are all classical normal subgroups of X.*

Proof. Similar to the proof of Theorem 3.6, therefore omitted.

Theorem 4.5. *Let X be a classical group and $A \in NNS(X)$. Let $X_A = \{x \in X \mid A(x) = A(e)\}$, where e is the unit of X. Then the classical subset X_A of X is a normal subgroup of X.*

Proof. Let $A \in NNS(X)$. First it is necessary to show that the classical set X_A is a subgroup of X. Let us take $x, y \in X_A$, then by Theorem 3.4

$$A(x \cdot y^{-1}) \geq A(x) \wedge A(y) = A(e) \wedge A(e) = A(e)$$

and always $A(e) \geq A(x \cdot y^{-1})$. Hence $x \cdot y^{-1} \in X_A$, i.e., X_A is a subgroup of X.

Now it will be shown that X_A is normal. Take arbitrary $x \in X_A$ and $y \in X$. Therefore, $A(x) = A(e)$. Since $A \in NNS(X)$, the following is obtained

$$A(y \cdot x \cdot y^{-1}) = A(y^{-1} \cdot y \cdot x) = A(x) = A(e).$$

Hence, $y \cdot x \cdot y^{-1} \in X_A$. So, X_A is a normal subgroup of X.

Theorem 4.6. *Let $g : X_1 \to X_2$ be a group homomorphism and $B \in NNS(X_2)$. Then the preimage $g^{-1}(B) \in NNS(X_1)$.*

Proof. From Theorem 3.8, it is known that $g^{-1}(B) \in NS(X_1)$. Hence it is sufficient to show that the normality property of $g^{-1}(B)$. For arbitrary $x_1, x_2 \in X_1$, by homomorphism of g and by the normality of B,

$$\begin{aligned} g^{-1}(B)(x_1 \cdot x_2) &= B(g(x_1 \cdot x_2)) \\ &= B(g(x_1) \cdot g(x_2)) \\ &= B(g(x_2) \cdot g(x_1)) \\ &= B(g(x_2 \cdot x_1)) \\ &= g^{-1}(B)(x_2 \cdot x_1). \end{aligned}$$

Hence, from Proposition 4.3, $g^{-1}(B) \in NNS(X_1)$.

Theorem 4.7. *Let $g : X_1 \to X_2$ be a surjective homomorphism of classical groups X_1 and X_2. If $A \in NNS(X_1)$, then $g(A) \in NNS(X_2)$.*

Proof. Since $g(A) \in NS(X_2)$ is clear from Theorem 3.7, it is sufficient only to show that the normality condition by using Proposition 4.3 (3). Take $y_1, y_2 \in X_2$ such that $g^{-1}(y_1) \neq \emptyset$, $g^{-1}(y_2) \neq \emptyset$ and $g^{-1}(y_1 \cdot y_2^{-1}) \neq \emptyset$. So it is inferred that

$$g(t_A)(y_1 \cdot y_2 \cdot y_1^{-1}) = \bigvee_{z \in g^{-1}(y_1 \cdot y_2 \cdot y_1^{-1})} t_A(z)$$
$$\text{and } g(t_A)(y_2) = \bigvee_{z \in g^{-1}(y_2)} t_A(z).$$

For all $x_2 \in g^{-1}(y_2), x_1 \in g^{-1}(y_1)$ and $x_1^{-1} \in g^{-1}(y_1^{-1})$, since A is normal,

$$t_A(x_1 \cdot x_2 \cdot x_1^{-1}) \geq t_A(x_2),$$
$$i_A(x_1 \cdot x_2 \cdot x_1^{-1}) \geq i_A(x_2) \text{ and}$$
$$f_A(x_1 \cdot x_2 \cdot x_1^{-1}) \leq f_A(x_2)$$

are obtained. Since g is a homomorphism, it follows that

$$g(x_1 \cdot x_2 \cdot x_1^{-1}) = g(x_1) \cdot g(x_2) \cdot g(x_1)^{-1}$$
$$= y_1 \cdot y_2 \cdot y_1^{-1}.$$

So, $x_1 \cdot x_2 \cdot x_1^{-1} \in g^{-1}(y_1 \cdot y_2 \cdot y_1^{-1})$. Hence

$$\bigvee_{z \in g^{-1}(y_1 \cdot y_2 \cdot y_1^{-1})} t_A(z)$$
$$\geq \bigvee_{x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2)} t_A(x_1 \cdot x_2 \cdot x_1^{-1})$$
$$\geq \bigvee_{x_2 \in g^{-1}(y_2)} t_A(x_2).$$

This means that

$$g(t_A)(y_1 \cdot y_2 \cdot y_1^{-1}) \geq g(t_A)(y_2).$$

On the other hand, the following inequalities are obtained in a similar way.

$$g(i_A)(y_1 \cdot y_2 \cdot y_1^{-1}) \geq g(i_A)(y_2) \text{ and}$$
$$g(f_A)(y_1 \cdot y_2 \cdot y_1^{-1}) \leq g(f_A)(y_2).$$

So the desired inequality,

$$g(A)(y_1 \cdot y_2 \cdot y_1^{-1})$$
$$= (g(t_A)(y_1 \cdot y_2 \cdot y_1^{-1}), g(i_A)(y_1 \cdot y_2 \cdot y_1^{-1}), g(f_A)(y_1 \cdot y_2 \cdot y_1^{-1}))$$
$$\geq (g(t_A)(y_2), g(i_A)(y_2), g(f_A)(y_2))$$
$$= (t_{g(A)}(y_2), i_{g(A)}(y_2), f_{g(A)}(y_2)) = g(A)(y_2).$$

is satisfied.

5. Conclusion

The concept of a group is of fundamental importance in the study of algebra. In order to study effectively an object with a given algebraic structure, it is necessary to study as well the functions that preserve the given algebraic structure (such functions are called homomorphisms). Normal subgroups play an important role in determining both the structure of a group X and the nature of homomorphisms with domain X. From this point of view, we decided to propose the definition of a neutrosophic subgroup and observed its fundamental properties. Also, we discussed normality of a neutrosophic subgroup of a classical group and studied its image and preimage under a group homomorphism. For further research one can handle cyclic (respectively, symmetric, abelian) neutrosophic group structure, and some of other algebraic structures such as ideal, ring, field etc. as well the neutrosophic topological structures.

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