Artificial Intelligence Review
https://doi.org/10.1007/s10462-022-10200-w

# Implementation of single-valued neutrosophic soft hypergraphs on human nervous system

**Muhammad Akram**[1] · **Hafiza Saba Nawaz**[1]



**Abstract**
Single-valued neutrosophic soft set simultaneously incorporates the attributes of both single-valued neutrosophic set as well as soft set. Corresponding to each parameter, it nominates a triplet $(\mathfrak{t}, \mathfrak{i}, \mathfrak{f})$ to a statement, where $\mathfrak{t}$, $\mathfrak{i}$ and $\mathfrak{f}$, respectively, describe the truthness, indeterminacy and falsity of that statement. In this article, we proceed in the framework of single-valued neutrosophic soft set by introducing single-valued neutrosophic soft hypergraphs which are effective to produce visual representation of connection among multiple objects of a system. Various fundamental operations such as union, join, Cartesian product and normal product of these graphical structures are suggested. We also discuss the construction of line graph and dual of single-valued neutrosophic soft hypergraphs with algorithms. The $r$-uniform single-valued neutrosophic soft hypergraphs with their operations like direct product, lexicographic product and costrong product is illustrated. In addition to this, we introduce the concept of regular, totally regular and perfectly regular single-valued neutrosophic soft hypergraphs and elaborate it with interesting results. Further, the single-valued neutrosophic soft directed hypergraphs together with some other interesting concepts have also been presented. At the end, it is explained that in what way, one can use the single-valued neutrosophic soft directed hypergraphs in the study of human nervous system. The proposed hypergraphs can be employed in artificial intelligence and decision-support systems effectively.

**Keywords** Single-valued neutrosophic soft sets · Hypergraphs · Directed hypergraphs · Human nervous system

## 1 Introduction

A hypergraph (Berge 1973, 1989) is an extension of graph whose edges (called hyperedges in this case) can link an arbitrary finite number of vertices. Each hypergraph can also be viewed as an incidence structure when studied in incidence geometry. A finite incidence

✉ Muhammad Akram
m.akram@pucit.edu.pk

Hafiza Saba Nawaz
sabanawaz707@gmail.com

1 Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

Published online: 20 May 2022

structure is an abstract system consisting of a set of points (vertices) and blocks (hyperedges) that are formed by the implementation of a sole relationship among points. This is the reason that a hypergraph is also represented as an incidence matrix. This discrete structure is quite strong as it is the most generalized approach to depict the multiple interactions among the objects of a system. If each vertex of a hypergraph is contained in $k$ hyperedges, then the hypergraph is called $k$-regular hypergraph. Additionally, for some fixed positive integer $r$, if all the hyperedges in a hypergraph link $r$ number of vertices then the hypergraph is called $r$-uniform hypergraph (Bretto 2013). Recently, tensor entropy (Chen and Rajapakse 2020), spectrum (Cardoso et al. 2020) and regularity (Liu et al. 2020) of uniform hypergraphs has been studied by different researchers. Hypergraphs have many applications in different areas such as the informatics and information systems, system modeling, social network analysis, system engineering, web information systems, service orientation architecture and much more (Molnár 2014). Various products of hypergraphs and specifically $r$-uniform hypergraphs were gathered and presented in Hellmuth et al. (2012).

Many a times, the classes of objects considered in actual-world do not possess exact boundaries. Based on this idea, Zadeh (1965) presented the notion of fuzzy set as a generalization of crisp set. It is characterized by a truth-membership function that designates the numerical value from the unit closed interval to every object of the considered class or universe. Up to now, a lot of work has been carried out in fuzzy set theory. Atanassov (1986) inserted another function called the falsity-membership function in fuzzy set and proposed the intuitionistic fuzzy set. This non-membership function provides the information about how much an object does not belongs to the considered set with a limitation that the summation of truth-membership and falsity-membership should not be greater than one. Based on the concept of neutrosophy, Smarandache (1998) approached the issue of uncertainty. He presented the neotrosophic set which is a broad context of crisp, fuzzy and intuitionistic fuzzy sets as it is characterized by three membership functions. To make this model applicable in real-world systems, Wang et al. (2010) put forth the single-valued neutrosophic set (SNS) and specified the framework of Smarandache's neutrosophic set from the scientific viewpoint. A SNS R on a non-empty space of points $V$ is defined by a 3-tuple of functions $\mathrm{R} = (\mathfrak{t}_{\mathrm{R}}, \mathfrak{i}_{\mathrm{R}}, \mathfrak{f}_{\mathrm{R}}) : V \rightarrow [0,1] \times [0,1] \times [0,1]$, where $\mathfrak{t}_{\mathrm{R}}$, $\mathfrak{i}_{\mathrm{R}}$ and $\mathfrak{f}_{\mathrm{R}}$, respectively, represent truth, indeterminacy and falsity membership functions. Numerous applications in information systems, decision support systems, information technology, medical diagnosis and applied physics were addressed by El-Hefenawy et al. (2016). Nguyen et al. (2019) presented a detailed description of SNSs in biomedical diagnosis. Many authors contributed in the study of SNSs as well as single-valued neutrosophic graphs in decision-making (Chutia and Smarandache 2021; Akram 2018; Sahin and Liu 2017; Mahapatra et al. 2021; Zeng et al. 2021; Karaaslan and Davvaz 2018; Peng et al. 2014; Yang et al. 2020).

Soft set ($\mathrm{S}_f\mathrm{S}$) theory (Molodtsov 1999) was proposed by Molodtsov in 1999 which addresses parameterized imprecision. Because of this theory, the use of parametrization has become very convenient as it permits functions, mappings, linguistic words or numerals as attributes. For if $A \subseteq \mathfrak{Z}$ denotes the set of parameters regarding all objects of universe $V$, a $\mathrm{S}_f\mathrm{S}$ is defined by the approximate mapping which produces a subset of universe for each $\mathfrak{z} \in A$. That is why, a soft set is also interpreted as a parameterized collection of subsets of universal set. The operations of soft sets are discussed in Maji et al. (2003); Sezgin and Atagün (2011). Fuzzy set and $\mathrm{S}_f\mathrm{S}$ deals with different types of imprecision namely, the membership and parameterized, respectively. Maji et al. (2001a) combined both these uncertainties in his work and named their model as fuzzy soft set. They also put forward the intuitionistic fuzzy soft set (Maji et al. 2001b) as an extension of fuzzy soft set. As the SNS explicitly handles the indeterminate information, Maji (2013) introduced the single-valued neutrosophic soft set ($SNS_fS$), discussed its operations and decision making application. Further, $SNS_f$ relations (Deli and Broumi 2015) were suggested by Deli and Broumi. This model is used by various researchers due to its aptness in numerous fields.

Hypergraphs were studied in fuzzy set theory by Mordeson and Nair (2001). They also discussed fuzzy transversals and coloring of these graphical structures. Parvathi et al. (2009) introduced the intuitionistic fuzzy hypergraphs and defined the dual and $(\alpha, \beta)$-cut of these hypergraphs. Afterwards, Akram et al. (2018) put forth single-valued neutrosophic hypergraph (SNH) and transversal SNH. Akram and Luqman (2017) discussed the applications of single-valued neutrosophic directed hypergraphs (SNDHs) in collaboration networks, production and manufacturing networks and social networks. Smarandache and Hassan (2016) investigated the regularity and completeness of SNH. The regularity, total regularity and uniformity of fuzzy soft hypergraphs was illustrated by Rashid et al. (2020). Intuitionistic fuzzy soft hypergraphs were proposed by Thilagavathi (2018). Shahzadi and Akram (2019) investigated Pythagorean fuzzy soft hypergraphs and discussed its regularity in detail. Akram and Luqman (2020) made worthwhile contribution to the studies of various extensions of hypergraphs. Table 1 shows the existing literature and its main findings.

We go ahead in $SNS_fS$ theory by introducing the single-valued neutrosophic soft hypergraphs ($SNS_fHs$) due to the following reasons:

1. $SNS_fS$ is a combination of SNS and $S_fS$. It can handle parameterized uncertainty with neutrosophic data which not only provides information about the membership and non-membership as compared to intuitionistic fuzzy set but also includes indeterminacy independently.
2. $SNS_fH$ carries feasible features as it can exhibit the interaction of multiple objects with its hyperedges pertaining to distinct attributes. This characteristic property of $SNS_fHs$ urged us to study them profoundly.

Motivated by the practicality of $SNS_fSs$ in various information systems as well as decision-making problems and, the capability of hypergraphs to represent the multiple relationships among the objects of a system, a new hybrid model of $SNS_fHs$ is established. This model includes the indeterminacy in a system accommodating multi-interactions to include neutralities found in real world. Different versions and operations of $SNS_fHs$ have been reported ahead in order to elaborate the proposed model. It is advantageous to use single-valued neutrosophic soft directed hypergraphs ($SNS_fDHs$) to represent the functionality of brain networks. This work partakes in the existing literature in the following way:

1. It proposes $SNS_fHs$ and suggests different types of its subhypergraphs. It presents union and join of two $SNS_fHs$. It also defines line graph and dual of a $SNS_fH$. It illustrates the concept of complete, strong and $r$-uniform $SNS_fHs$ as well as different products of $SNS_fHs$.
2. It also explains the idea of $SNS_fDHs$ with examples. It presents the aptness of the proposed model in the studies of human nervous system.

The arrangement of article is as follows. Section 2 provides the preliminary work corresponding to this manuscript. Section 3 gives brief description of $SNS_fHs$. The next

**Table 1** Existing literature

| Authors | Proposed work | Main findings |
|---|---|---|
| Wang et al. (2010) | Introduction to SNS | 1. Proposed SNS as an instance of Smarandach's neutrosophic set and also studied the set-theoretic properties of SNS |
| Molnár (2014) | Applications of hypergraphs | 1. Revealed numerous applications of hypergraphs in the fields of information technology, information system and decision-support system |
| Maji (2013) | $SNS_fS$ | 1. Combined SNS and soft set to give rise a new mathematical model namely, $SNS_fS$ and illustrated its various operations |
| Hellmuth et al. (2012) | Survey on hypergraph products | 1. Presented survey on hypergraph products which are the generalizations of standard graph products<br>2. Provided corresponding results on products of finite (directed as well as undirected) and infinite hypergraphs |
| Akram and Luqman (2017) | Network models of SNDH | 1. Applied the concept of SNS to directed hypergraphs<br>2. Described the aptness of SNDH in social, production and collaboration networks |
| Akram et al. (2018) | SNHs | 1. Presented SNH, line graph as well as dual of SNH<br>2. Investigated several results supporting the proposed hypergraph model |

section is followed by the study of $r$-uniform SNS$_f$Hs. Section 5 explains the concept of regularity for the proposed hybrid model of hypergraphs. In Sect. 6, the SNS$_f$DHs are introduced. Section 7 presents the implementation of SNS$_f$DHs on the networks of human nervous system studies and last section gives the concluding remarks of this work.

## 2 Preliminaries

In this section, we give some preliminary concepts that will support to apprehend further research work.

**Definition 1** Berge (1973, 1989) A simple hypergraph $H$ is denoted by the pair $H = (V, E)$, where $V = \{v_i : 1 \le i \le n\}$ is a non-void finite set of objects called vertices/nodes and $E$ is a subset of $P(V)\backslash\{\phi\}$ ($P(V)$ represents the power set of V). The members $E_j = \{v_k : 1 \le k \le m, 2 \le m \le n\} (1 \le j \le t)$ of $E$ are subsets of $V$ and are known as hyperedges of $H$. The cardinality of vertex set $|V|$ and that of hyperedge set $|E|$ is called the order $O(H)$ and size $S(H)$ of $H$, respectively.

The number of hyperedges that contain vertex $v$ is known as degree $d(v)$ of that vertex. If each vertex of $H$ is contained in $k$ number of hyperedges, i.e., $d(v) = k$, $\forall v \in V$, then $H$ is called $k$-regular hypergraph. Moreover, if the hyperedges of $H$ contain equal number of vertices in them, let it be $r$, then $H$ is called $r$-uniform hypergraph (Bretto 2013).

**Definition 2** Wang et al. (2010) A SNS R on a non-empty space of points $V$ is defined by a 3-tuple of functions $\mathrm{R} = (\mathfrak{t}_{\mathrm{R}}, \mathfrak{i}_{\mathrm{R}}, \mathfrak{f}_{\mathrm{R}}) : V \to [0,1] \times [0,1] \times [0,1]$, where $\mathfrak{t}_{\mathrm{R}}$, $\mathfrak{i}_{\mathrm{R}}$ and $\mathfrak{f}_{\mathrm{R}}$, respectively, represent truth, indeterminacy and falsity membership functions.

**Definition 3** Maji (2013) Let $V$ be the space of points, $A$ be the set of parameters and $\mathcal{P}(V)$ denotes the set of all SNSs over $V$. The single-valued neutrosophic soft set $(\mathrm{R}, A)$ is the parameterized collection of SNSs, defined by the approximate mapping $\mathrm{R} : A \to \mathcal{P}(V)$.

**Definition 4** Akram and Shahzadi (2017) Let $V$ be a non-void set of objects. A single-valued neutrosophic soft graph G is denoted by the tuple $\mathrm{G} = (\mathrm{C}, \mathrm{D}, A)$, where $(\mathrm{C}, A)$ is the SNS$_f$S of vertices and $(\mathrm{C}, A)$ is the SNS$_f$S of edges. Additionally, $\mathrm{G}(\mathfrak{z}) = (\mathrm{C}(\mathfrak{z}), \mathrm{D}(\mathfrak{z}))$ is the single-valued neutrosophic graph corresponding to parameter $\mathfrak{z} \in A$ such that

$$\begin{aligned}
\mathfrak{t}_{\mathrm{C}(\mathfrak{z})}(v_i v_j) &\le \mathfrak{t}_{\mathrm{D}(\mathfrak{z})}(v_i) \wedge \mathfrak{t}_{\mathrm{D}(\mathfrak{z})}(v_j)\\
\mathfrak{i}_{\mathrm{C}(\mathfrak{z})}(v_i v_j) &\le \mathfrak{i}_{\mathrm{D}(\mathfrak{z})}(v_i) \wedge \mathfrak{i}_{\mathrm{D}(\mathfrak{z})}(v_j)\\
\mathfrak{f}_{\mathrm{C}(\mathfrak{z})}(v_i v_j) &\le \mathfrak{f}_{\mathrm{D}(\mathfrak{z})}(v_i) \vee \mathfrak{f}_{\mathrm{D}(\mathfrak{z})}(v_j)
\end{aligned}$$

for all $v_i, v_j \in V$.

## 3 Single-valued neutrosophic soft hypergraph

**Definition 5** Let $H = (V, E)$ denotes a crisp hypergraph. A SNS$_f$H H over $H$ is denoted by the ordered triplet $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$, where $A$ denotes the set of parameters and

(1) (R, $A$) is a SNS$_f$S of vertices over $V$ such that R : $A \to \mathcal{P}(V)$ is a SNS$_f$ approximate mapping ($\mathcal{P}(V)$ denotes the set of all SNSs over $V$).
(2) (S, $A$) is a SNS$_f$S over $E(\subseteq V^m$, $m$ being the finite positive integer) and S : $A \to \mathcal{P}(E)$ is the corresponding SNS$_f$ mapping such that the member $E_j(1 \le j \le t)$ of S($\mathfrak{z}$) represents the SN hyperedge in the SNH H($\mathfrak{z}$) = (R($\mathfrak{z}$), S($\mathfrak{z}$)) of H, and its truth-membership, indeterminacy membership and falsity-membership values can be computed as

$$\mathfrak{t}_{S(\mathfrak{z})}(E_j) = \mathfrak{t}_{S(\mathfrak{z})}(v_1v_2...v_m) \le \min\{\mathfrak{t}_{R(\mathfrak{z})}(v_1), \mathfrak{t}_{R(\mathfrak{z})}(v_2), ..., \mathfrak{t}_{R(\mathfrak{z})}(v_m)\},$$
$$\mathfrak{i}_{S(\mathfrak{z})}(E_j) = \mathfrak{i}_{S(\mathfrak{z})}(v_1v_2...v_m) \le \min\{\mathfrak{i}_{R(\mathfrak{z})}(v_1), \mathfrak{i}_{R(\mathfrak{z})}(v_2), ..., \mathfrak{i}_{R(\mathfrak{z})}(v_m)\},$$
$$\mathfrak{f}_{S(\mathfrak{z})}(E_j) = \mathfrak{f}_{S(\mathfrak{z})}(v_1v_2...v_m) \le \max\{\mathfrak{f}_{R(\mathfrak{z})}(v_1), \mathfrak{f}_{R(\mathfrak{z})}(v_2), ..., \mathfrak{f}_{R(\mathfrak{z})}(v_m)\},$$

respectively, where $2 \le m \le n$.
(3) For all parameters $\mathfrak{z}$, $\bigcup_{1 \le j \le t} Supp(E_j) = V$, where $E_j$ denotes the SN hyperedge in $H(\mathfrak{z})$ .

From now on, we denote $|E_j| = \varepsilon_j$, where $E_j \in E$.

**Definition 6** The order $\mathcal{O}$(H) of a SNS$_f$H H = (R, S, $A$) is defined as

$$\mathcal{O}(\mathrm{H}) = \sum_{\mathfrak{z}\in A}\Big(\sum_{v\in V} \mathfrak{t}_{R(\mathfrak{z})}(v), \sum_{v\in V} \mathfrak{i}_{R(\mathfrak{z})}(v), \sum_{v\in V} \mathfrak{f}_{R(\mathfrak{z})}(v)\Big).$$

The size $\mathcal{S}$(H) of a SNS$_f$H H = (R, S, $A$) is defined as

$$\mathcal{S}(\mathrm{H}) = \sum_{\mathfrak{z}\in A}\Big(\sum_{j} \mathfrak{t}_{S(\mathfrak{z})}(E_j), \sum_{j} \mathfrak{i}_{S(\mathfrak{z})}(E_j), \sum_{j} \mathfrak{f}_{S(\mathfrak{z})}(E_j)\Big).$$

**Definition 7** The strength $\eta$ of a SNS$_f$ hyperedge $E_j$ in a SNS$_f$H H is defined as

$$\eta(E_j) = (\min_{\mathfrak{z}} \min_{v_k\in E_j} \mathfrak{t}_{R(\mathfrak{z})}(v_k), \min_{\mathfrak{z}} \min_{v_k\in E_j} \mathfrak{i}_{R(\mathfrak{z})}(v_k), \max_{\mathfrak{z}} \max_{v_k\in E_j} \mathfrak{f}_{R(\mathfrak{z})}(v_k)).$$

***Example 1*** Consider a SNS$_f$H H = (R, S, $A$) over $H = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_1v_2\,v_4v_5, v_1v_3v_5, v_2v_3v_4, v_4v_5\}$ such that

$$\begin{aligned}
\mathrm{H}(\mathfrak{z}_1) &= (\mathrm{R}(\mathfrak{z}_1), \mathrm{S}(\mathfrak{z}_1)) \\
&= (\{\langle v_1, (0.9, 0.5, 0.7)\rangle, \langle v_2, (0.8, 0.6, 1.0)\rangle, \langle v_3, (0.6, 0.7, 0.9)\rangle, \langle v_4, (0.7, 0.7, 0.8)\rangle, \langle v_5, (0.8, \\
&\quad 0.6, 0.4)\rangle\}, \{\langle v_1v_2, (0.7, 0.4, 0.8)\rangle, \langle v_1v_3v_5, (0.6, 0.3, 0.8)\rangle, \langle v_2v_3v_4, (0.5, 0.5, 0.9)\rangle, \langle v_4v_5, (0.4, 0.3, 0.7)\rangle\}), \\
\mathrm{H}(\mathfrak{z}_2) &= (\mathrm{R}(\mathfrak{z}_2), \mathrm{S}(\mathfrak{z}_2)) \\
&= (\{\langle v_1, (0.7, 0.9, 0.8)\rangle, \langle v_2, (0.9, 0.6, 0.7)\rangle, \langle v_3, (0.8, 0.7, 0.5)\rangle, \langle v_4, (1.0, 0.5, 0.8)\rangle, \langle v_5, (0.7, \\
&\quad 0.9, 0.8)\rangle\}, \{\langle v_1v_2v_4v_5, (0.8, 0.4, 0.6)\rangle, \langle v_1v_3v_5, (0.6, 0.7, 0.5)\rangle, \langle v_2v_3v_4, (0.7, 0.3, 0.8)\rangle\}).
\end{aligned}$$

Figure 1 displays the corresponding SNS$_f$H.

The order and size of H are $\mathcal{O}(\mathrm{H}) = (7.9, 6.7, 7.4)$ and $\mathcal{S}(\mathrm{H}) = (4.3, 2.9, 5.1)$, respectively. Also, $\eta(v_1v_3v_5) = (0.6, 0.5, 0.9)$ is the strength of a SNS$_f$ hyperedge in H.

**Definition 8** A SNS$_f$H H′ = (R′, S′, $A'$) is said to be a SNS$_f$ subhypergraph of H = (R, S, $A$) if

1. $A' \subseteq A$,
2. H′($\mathfrak{z}$) is a partial SN subhypergraph of H($\mathfrak{z}$), $\forall \mathfrak{z} \in A'$, i.e., R′($\mathfrak{z}$) ⊆ R($\mathfrak{z}$) and S′($\mathfrak{z}$) ⊆ S($\mathfrak{z}$).

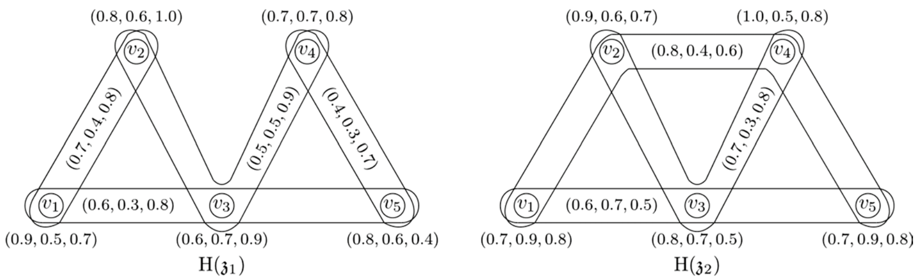



**Fig. 1** A $\mathrm{SNS}_f\mathrm{H}$ H

***Example 2*** Consider the $\mathrm{SNS}_f\mathrm{H}$ H given in Fig. 1. The $\mathrm{SNS}_f$ subhypergraph H′ of H is shown in Fig. 2.

**Definition 9** A $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H}' = (\mathrm{R}', \mathrm{S}', A')$ is said to be a spanning $\mathrm{SNS}_f$ subhypergraph of $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ if

1. $A' \subseteq A$,
2. $\mathrm{R}'(\mathfrak{z}) = \mathrm{R}(\mathfrak{z})$, for all $\mathfrak{z} \in A'$.

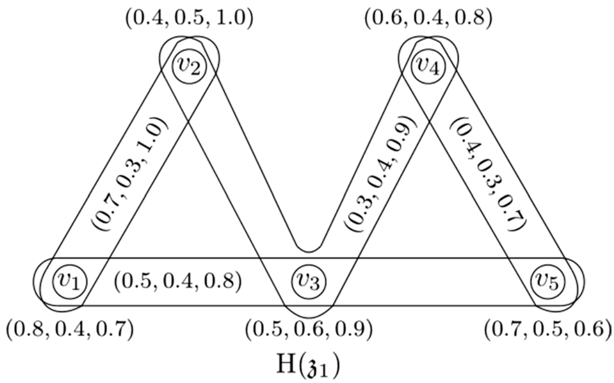
In this case, the two $\mathrm{SNS}_f\mathrm{H}$s have same $\mathrm{SNS}_f$ vertex set, they differ only in the neutrosophic grades of hyperedges.

***Example 3*** Consider the $\mathrm{SNS}_f\mathrm{H}$ H given in Fig. 1. The spanning $\mathrm{SNS}_f$ subhypergraph H′ of H is shown in Fig. 3.

**Definition 10** Let $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ be a $\mathrm{SNS}_f\mathrm{H}$ over $H = (V, E)$. A $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H}' = (\mathrm{R}', \mathrm{S}', A')$ over $H' = (V', E')$ is said to be $\mathrm{SNS}_f$ subhypergraph of H induced by $(\mathrm{R}', A')$ if

1. $A' \subseteq A$,
2. $\mathrm{R}'(\mathfrak{z}) \subseteq \mathrm{R}(\mathfrak{z})$ over $V' \subseteq V$, for all $\mathfrak{z} \in A'$,
3. $\mathrm{S}'(\mathfrak{z})$ is defined over $E' = \{E'_j = E_i \cap V' \neq \phi : E_i \in E and |E_i \cap V'| \geq 2\}$ such that

**Fig. 2** A $\mathrm{SNS}_f$ subhypergraph H′



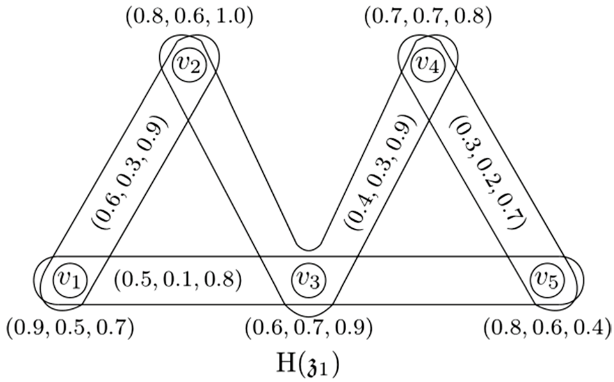



**Fig. 3** A spanning $\mathrm{SNS}_f$ subhypergraph $\mathrm{H}'$

$$\mathfrak{t}_{\mathrm{S}'(\mathfrak{z})}(E'_j) = \mathfrak{t}_{\mathrm{S}'(\mathfrak{z})}(v_1 v_2 ... v_m) = \mathfrak{t}_{\mathrm{R}'(\mathfrak{z})}(v_1) \wedge \mathfrak{t}_{\mathrm{R}'(\mathfrak{z})}(v_2) \wedge ... \wedge \mathfrak{t}_{\mathrm{R}'(\mathfrak{z})}(v_m) \wedge \mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_i),$$
$$\mathfrak{i}_{\mathrm{S}'(\mathfrak{z})}(E'_j) = \mathfrak{t}_{\mathrm{S}'(\mathfrak{z})}(v_1 v_2 ... v_m) = \mathfrak{i}_{\mathrm{R}'(\mathfrak{z})}(v_1) \wedge \mathfrak{i}_{\mathrm{R}'(\mathfrak{z})}(v_2) \wedge ... \wedge \mathfrak{i}_{\mathrm{R}'(\mathfrak{z})}(v_m) \wedge \mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_i),$$
$$\mathfrak{f}_{\mathrm{S}'(\mathfrak{z})}(E'_j) = \mathfrak{f}_{\mathrm{S}'(\mathfrak{z})}(v_1 v_2 ... v_m) = \mathfrak{f}_{\mathrm{R}'(\mathfrak{z})}(v_1) \vee \mathfrak{f}_{\mathrm{R}'(\mathfrak{z})}(v_2) \vee ... \vee \mathfrak{f}_{\mathrm{R}'(\mathfrak{z})}(v_m) \vee \mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(E_i).$$

***Example 4*** Consider the $\mathrm{SNS}_f$H H given in Fig. 4a. The $\mathrm{SNS}_f$ subhypergraph $\mathrm{H}'$ of H induced by $(\mathrm{R}', A')$ is shown in Fig. 4b.

**Definition 11** Let $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ be a $\mathrm{SNS}_f$H over $H = (V, E)$. A SN hyperpath $\mathrm{P}(\mathfrak{z})(v_1, v_p)$ from $v_1$ to $v_p$ in $\mathrm{H}(\mathfrak{z})$ for some $\mathfrak{z} \in A$ is defined as an alternative sequence $v_1 E_1 v_2 E_2 ... v_{p-1} E_{p-1} v_p$ of distinct vertices and hyperedges such that

- $v_i, v_{i+1} \in E_i$, and
- at least one of the truth-membership, indeterminacy membership and falsity-membership values is non-zero for all vertices and hyperedges of $\mathrm{P}(\mathfrak{z})(v_1, v_p)$.

The integer $p - 1$ is called the length of $\mathrm{P}(\mathfrak{z})(v_1, v_p)$. If $\mathrm{P}(\mathfrak{z})(v_1, v_p)$ is a SN hyperpath, $\forall \mathfrak{z}$, then $v_1 E_1 v_2 E_2 ... v_{p-1} E_{p-1} v_p$ is called a $\mathrm{SNS}_f$ hyperpath and is denoted by $\mathrm{P}(v_1, v_p)$. Further, If $v_1 = v_p$, then the $\mathrm{SNS}_f$ hyperpath $\mathrm{P}(v_1, v_p)$ is called $\mathrm{SNS}_f$ hypercycle C.

**Definition 12** Let $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ be a $\mathrm{SNS}_f$H over $H = (V, E)$. The SNH $\mathrm{H}(\mathfrak{z})$ in H is called connected if there exists at least one SN hyperpath $\mathrm{P}(\mathfrak{z})(v_i, v_j)$ for each pair of distinct vertices $v_i, v_j$ in $\mathrm{H}(\mathfrak{z})$. Moreover, if $\mathrm{H}(\mathfrak{z})$ is connected SNH for all $\mathfrak{z}$, then H is called connected $\mathrm{SNS}_f$H.

**Definition 13** A $\mathrm{SNS}_f$H $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ over $H = (V, E)$ is said to be a strong $\mathrm{SNS}_f$H if $\mathrm{H}(\mathfrak{z})$ is a strong SN hypergraph for all $\mathfrak{z} \in A$, i.e.,

$$\mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j) = \mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(v_1 v_2 ... v_m) = \min\{\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v_1), \mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v_2), ..., \mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v_m)\},$$
$$\mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j) = \mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(v_1 v_2 ... v_m) = \min\{\mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v_1), \mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v_2), ..., \mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v_m)\},$$
$$\mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(E_j) = \mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(v_1 v_2 ... v_m) = \max\{\mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v_1), \mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v_2), ..., \mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v_m)\},$$

for all hyperedges $E_j \in E$.

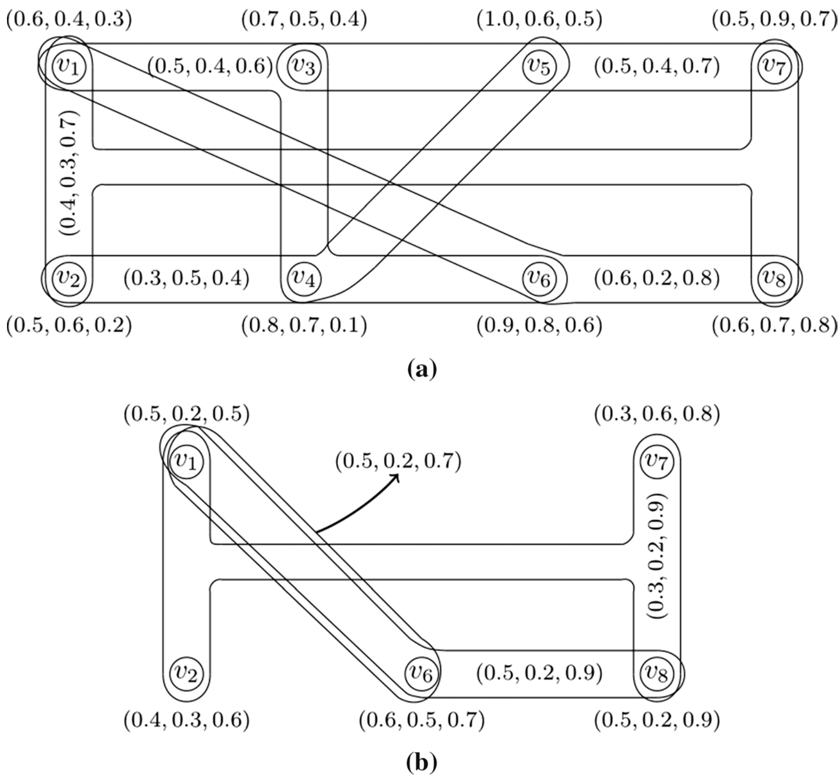



**Fig. 4** **a** A $SNS_f$H H. **b** A $SNS_f$ subhypergraph H′ of H induced by $SNS_f$ vertex set of H′

***Example 5*** Consider the $SNS_f$H H = (R, S, $A$) given in Fig. 5. Clearly, it is a strong $SNS_f$H.

**Definition 14** A $SNS_f$H H = (R, S, $A$) over $H = (V, E)$ is said to be a complete $SNS_f$H if H($\mathfrak{z}$) is a complete SN hypergraph for all $\mathfrak{z} \in A$, i.e., if for all parameters $\mathfrak{z}$, $E = P(V) \setminus \{\emptyset\}$ such that

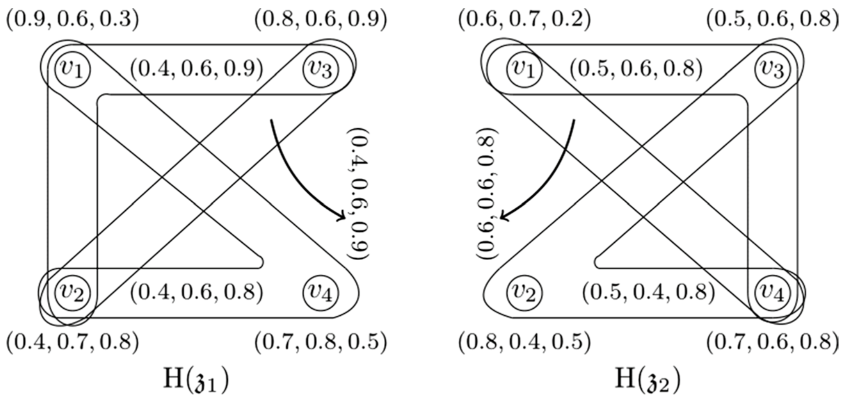



**Fig. 5** A strong $SNS_f$H

$$\mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j) = \mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(v_1v_2...v_m) = \min\{\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v_1), \mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v_2), ..., \mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v_m)\},$$
$$\mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j) = \mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(v_1v_2...v_m) = \min\{\mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v_1), \mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v_2), ..., \mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v_m)\},$$
$$\mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(E_j) = \mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(v_1v_2...v_m) = \max\{\mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v_1), \mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v_2), ..., \mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v_m)\}.$$

***Example 6*** Consider the $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ given in Fig. 6. Clearly, it is a complete $\mathrm{SNS}_f\mathrm{H}$ without loops.

**Definition 15** The union of two $\mathrm{SNS}_f\mathrm{H}$s $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ and $\mathrm{H}' = (\mathrm{R}', \mathrm{S}', A')$ over $H = (V, E)$ and $H' = (V', E')$, respectively, is a $\mathrm{SNS}_f\mathrm{H}$. It can be represented as $\mathrm{H} \cup \mathrm{H}' = (\mathrm{R} \cup \mathrm{R}', \mathrm{S} \cup \mathrm{S}', A \cup A')$ , where $(\mathrm{R} \cup \mathrm{R}', A \cup A')$ is a $\mathrm{SNS}_f\mathrm{S}$ of vertices over $V \cup V'$ and $(\mathrm{S} \cup \mathrm{S}', A \cup A')$ is a $\mathrm{SNS}_f\mathrm{S}$ of hyperedges over $E \cup E'$ and $(\mathrm{H} \cup \mathrm{H}')(\mathfrak{z}) = ((\mathrm{R} \cup \mathrm{R}')(\mathfrak{z}), (\mathrm{S} \cup \mathrm{S}')(\mathfrak{z}))$ is a SN hypergraph for all $\mathfrak{z} \in A \cup A'$ defined by

$$(\mathrm{H} \cup \mathrm{H}')(\mathfrak{z}) = \begin{cases} \mathrm{H}(\mathfrak{z}), & if\, \mathfrak{z} \in A - A', \\ \mathrm{H}'(\mathfrak{z}), & if\, \mathfrak{z} \in A' - A, \\ \mathrm{H}(\mathfrak{z}) \cup \mathrm{H}'(\mathfrak{z}), & if\, \mathfrak{z} \in A \cap A', \end{cases}$$

where $\mathrm{H}(\mathfrak{z}) \cup \mathrm{H}'(\mathfrak{z})$ denotes the union of $\mathrm{H}(\mathfrak{z})$ and $\mathrm{H}'(\mathfrak{z})$ for all $\mathfrak{z} \in A \cap A'$.

***Remark 1*** In above definition, if $V$ and $V'$ are disjoint sets then $\mathrm{H} \cup \mathrm{H}'$ is called disjoint union of H and H′.

***Example 7*** Consider a $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ over $H = (V, E)$, where $A = \{\mathfrak{z}_1, \mathfrak{z}_2\}$, $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_1v_2v_3, v_1v_2v_4, v_1v_3v_4, v_3v_4\}$ given in Fig. 7.

Consider another $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H}' = (\mathrm{R}', \mathrm{S}', A')$ over $H' = (V', E')$, where $A' = \{\mathfrak{z}_1, \mathfrak{z}_2\}$, $V' = \{v_1, v_2, v_3, v_4, v_5\}$ and $E' = \{v_1v_2, v_1v_2v_4, v_1v_3v_5, v_2v_4v_5, v_4v_5\}$ given in Fig. 8.

The union $\mathrm{H} \cup \mathrm{H}'$ of H and H′ is presented in Fig. 9.

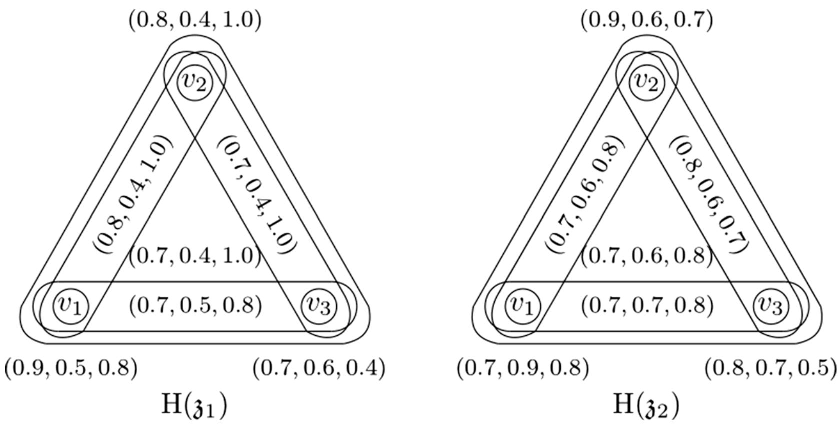



**Fig. 6** A complete $\mathrm{SNS}_f\mathrm{H}$

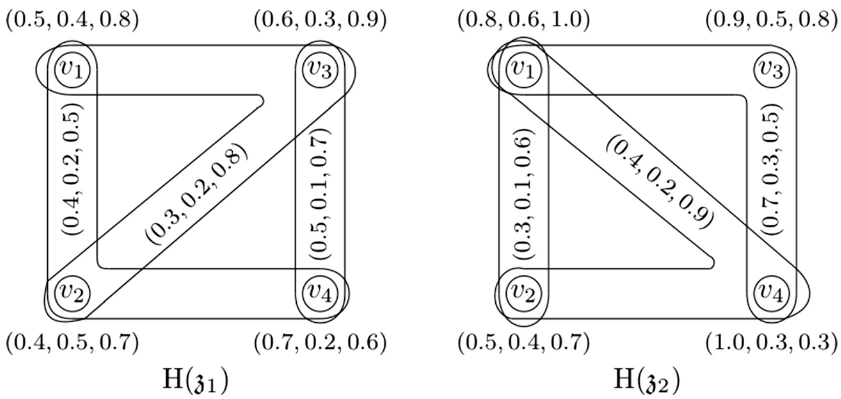



**Fig. 7** A $SNS_f$H H

**Proposition 1** *Let* $\mathrm{H} \cup \mathrm{H}'$ *be the union of two* $SNS_f$*Hs* H *and* H′, *then* H *and* H′ *are the* $SNS_f$ *subhypergraphs of* $\mathrm{H} \cup \mathrm{H}'$.

**Definition 16** The join of two $SNS_f$Hs $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ and $\mathrm{H}' = (\mathrm{R}', \mathrm{S}', A')$ over $H = (V, E)$ and $H' = (V', E')$, respectively, is a $SNS_f$H. It can be represented as $\mathrm{H} + \mathrm{H}' = (\mathrm{R} + \mathrm{R}', \mathrm{S} + \mathrm{S}', A \cup A')$ , where $(\mathrm{R} + \mathrm{R}', A \cup A')$ is a $SNS_f$S of vertices over $V \cup V'$ and $(\mathrm{S} + \mathrm{S}', A \cup A')$ is a $SNS_f$S of hyperedges over $E \cup E' \cup E^+$, where $E^+$ is the set of all edges joining the vertices in $V$ and $V'$. Further, it is assumed that $V \cap V' = \phi$ and $(\mathrm{H} + \mathrm{H}')(\mathfrak{z}) = ((\mathrm{R} + \mathrm{R}')(\mathfrak{z}), (\mathrm{S} + \mathrm{S}')(\mathfrak{z}))$ is a SN hypergraph for all $\mathfrak{z} \in A \cup A'$ defined by

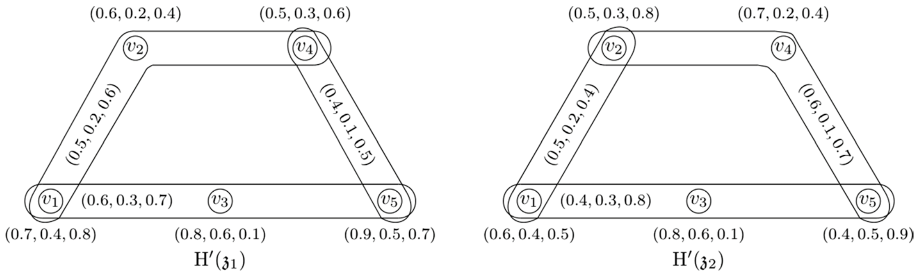



**Fig. 8** A $SNS_f$H H′

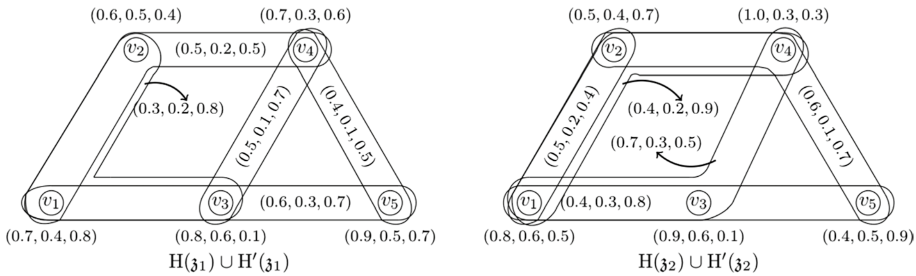



**Fig. 9** A $SNS_f$H $\mathrm{H} \cup \mathrm{H}'$

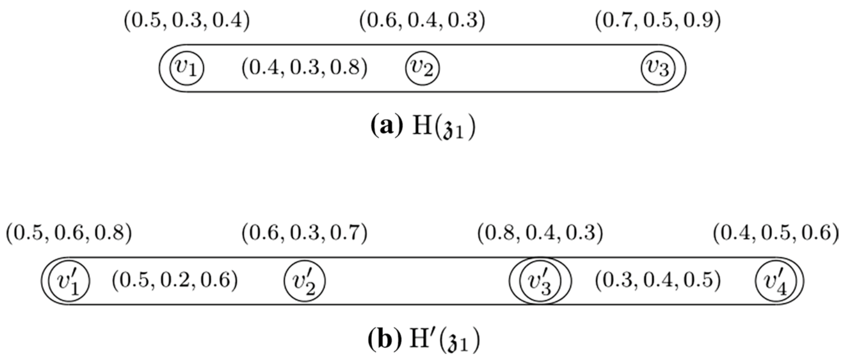



**Fig. 10** **a** A SNS$_f$H H **b** A SNS$_f$H H′

$$(\mathrm{H}+\mathrm{H}')(\mathfrak{z}) = \begin{cases} \mathrm{H}(\mathfrak{z}), & if\, \mathfrak{z} \in A - A', \\ \mathrm{H}'(\mathfrak{z}), & if\, \mathfrak{z} \in A' - A, \\ \mathrm{H}(\mathfrak{z}) + \mathrm{H}'(\mathfrak{z}), & if\, \mathfrak{z} \in A \cap A', \end{cases}$$

where $\mathrm{H}(\mathfrak{z}) + \mathrm{H}'(\mathfrak{z})$ denotes the join of $\mathrm{H}(\mathfrak{z})$ and $\mathrm{H}'(\mathfrak{z})$ for all $\mathfrak{z} \in A \cap A'$.

***Example 8*** Consider a SNS$_f$H $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ over $H = (V, E)$, where $A = \{\mathfrak{z}_1\}$, $V = \{v_1, v_2, v_3\}$ and $E = \{v_1 v_2 v_3\}$, whose graphical representation is given in Fig. 10a. Consider another SNS$_f$H $\mathrm{H}' = (\mathrm{R}', \mathrm{S}', A')$ over $H' = (V', E')$, where $A' = \{\mathfrak{z}_1\}$, $V' = \{v'_1, v'_2, v'_3\}$ and $E' = \{v'_1 v'_2 v'_3, v'_3 v'_4\}$, which is presented graphically in Fig. 10b.

The join $\mathrm{H} + \mathrm{H}'$ of the considered SNS$_f$Hs H and H′ is given by

$$\begin{aligned} \mathrm{H} + \mathrm{H}' =& \mathrm{H}(\mathfrak{z}_1) + \mathrm{H}'(\mathfrak{z}_1) = (\mathrm{R}(\mathfrak{z}_1) + \mathrm{R}'(\mathfrak{z}_1), \mathrm{S}(\mathfrak{z}_1) + \mathrm{S}'(\mathfrak{z}_1)) = (\{\langle v_1, (0.5, 0.3, 0.4)\rangle, \langle v_2, (0.6, 0.4, 0.3)\rangle, \langle v_3, (0.7, \\ & 0.5, 0.9)\rangle, \langle v'_1, (0.5, 0.6, 0.8)\rangle, \langle v'_2, (0.6, 0.3, 0.7)\rangle, \langle v'_3, (0.8, 0.4, 0.3)\rangle, \langle v'_4, (0.4, 0.5, 0.6)\rangle\}, \{\langle v_1 v_2 v_3, (0.4, \\ & 0.3, 0.8)\rangle, \langle v'_1 v'_2 v'_3, (0.5, 0.2, 0.6)\rangle, \langle v'_3 v'_4, (0.3, 0.4, 0.5)\rangle, \langle v_1 v'_1, (0.5, 0.3, 0.8)\rangle, \langle v_1 v'_2, (0.5, 0.3, 0.7)\rangle, \langle v_1 \\ & v'_3, (0.5, 0.3, 0.4)\rangle, \langle v_1 v'_4, (0.4, 0.3, 0.6)\rangle, \langle v_2 v'_1, (0.5, 0.4, 0.8)\rangle, \langle v_2 v'_2, (0.6, 0.3, 0.7)\rangle, \langle v_2 v'_3, (0.6, 0.4, 0.3)\rangle, \\ & \langle v_2 v'_4, (0.4, 0.4, 0.6)\rangle, \langle v_3 v'_1, (0.5, 0.5, 0.9)\rangle, \langle v_3 v'_2, (0.6, 0.3, 0.9)\rangle, \langle v_3 v'_3, (0.7, 0.4, 0.9)\rangle, \langle v_3 v'_4, (0.4, 0.5, \\ & 0.9)\rangle\}). \end{aligned}$$

The SNS$_f$H $\mathrm{H} + \mathrm{H}'$ is presented graphically in Fig. 11.

**Definition 17** Let $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ be a SNS$_f$H over $H = (V, E)$. The line graph $\mathcal{L}(\mathrm{H}) = (\mathrm{R}^l, \mathrm{S}^l, A)$ of H is, in fact, the collection of line graphs $\mathcal{L}(\mathrm{H}(\mathfrak{z}))$ of $\mathrm{H}(\mathfrak{z})$, for all $\mathfrak{z}$. The line graph $\mathcal{L}(\mathrm{H}(\mathfrak{z}))$ over $H^l = (V^l, E^l)$ is defined by considering $\mathrm{R}^l(\mathfrak{z}) = \mathrm{S}(\mathfrak{z})$ over $V^l = \{V_j = E_j : E_j \in E\}$ and $\mathrm{S}^l(\mathfrak{z})$ over $E^l = \{V_p V_q : V_p \cap V_q \neq \emptyset\}$ such that for each SN edge between two non-trivial SN vertices $V_p$ and $V_q$ of $\mathcal{L}(\mathrm{H}(\mathfrak{z}))$, the neutrosophic grades are

$$\begin{aligned} \mathfrak{t}_{\mathrm{S}^l(\mathfrak{z})}(V_p V_q) &= \mathfrak{t}_{\mathrm{R}^l(\mathfrak{z})}(V_p) \wedge \mathfrak{t}_{\mathrm{R}^l(\mathfrak{z})}(V_q), \\ \mathfrak{i}_{\mathrm{S}^l(\mathfrak{z})}(V_p V_q) &= \mathfrak{i}_{\mathrm{R}^l(\mathfrak{z})}(V_p) \wedge \mathfrak{i}_{\mathrm{R}^l(\mathfrak{z})}(V_q), \\ \mathfrak{f}_{\mathrm{S}^l(\mathfrak{z})}(V_p V_q) &= \mathfrak{f}_{\mathrm{R}^l(\mathfrak{z})}(V_p) \vee \mathfrak{f}_{\mathrm{R}^l(\mathfrak{z})}(V_q). \end{aligned}$$

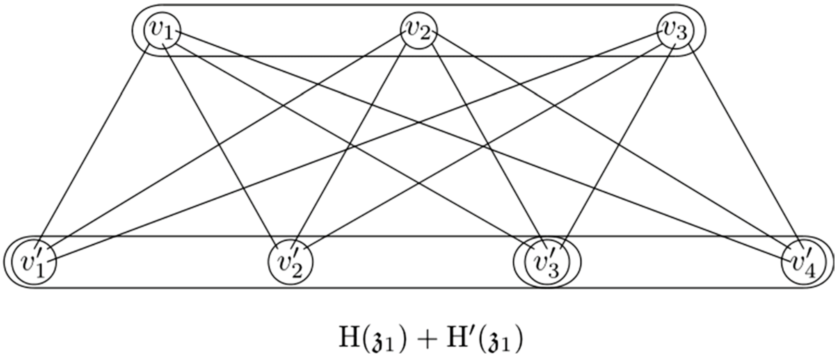



**Fig. 11** A $SNS_f$H H + H′

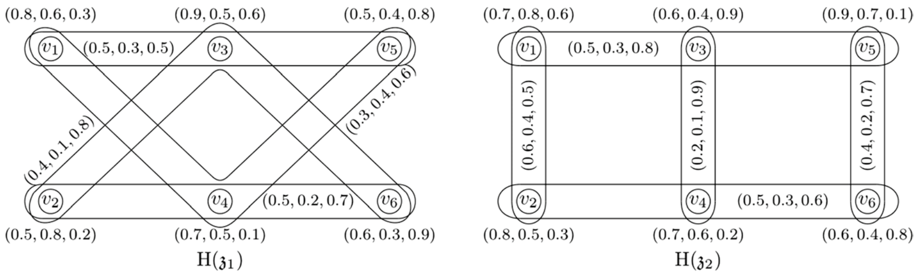



**Fig. 12** A $SNS_f$H H

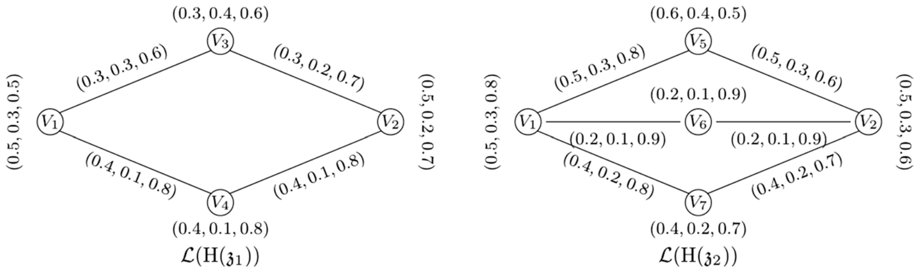



**Fig. 13** A line graph $\mathcal{L}$(H) of H

***Remark 2*** The above definition clearly shows that the line graph of a $SNS_f$H is an intersection graph of that $SNS_f$H.

Algorithm 31 gives the procedure of the construction of line graph of $SNS_f$H.

**Algorithm 31** *Line graph of $SNS_f H$*

**Input**: *A* $SNS_f H$ $\mathrm{H} = \{\mathrm{H}(\mathfrak{z}_r) : 1 \leq r \leq s, \mathfrak{z}_r \in A\}$ *over* $H = (V, E)$, *where* $\mathrm{H}(\mathfrak{z}_r) = (\mathrm{R}(\mathfrak{z}_r), \mathrm{S}(\mathfrak{z}_r))$,
$\mathrm{R}(\mathfrak{z}_r) = \{\langle v_i, (\mathfrak{t}_{\mathrm{R}(\mathfrak{z}_r)}(v_i), \mathfrak{i}_{\mathrm{R}(\mathfrak{z}_r)}(v_i), \mathfrak{f}_{\mathrm{R}(\mathfrak{z}_r)}(v_i))\rangle : 1 \leq i \leq n, v_i \in V\}$ *and*
$\mathrm{S}(\mathfrak{z}_r) = \{\langle E_j, (\mathfrak{t}_{\mathrm{S}(\mathfrak{z}_r)}(E_j), \mathfrak{i}_{\mathrm{S}(\mathfrak{z}_r)}(E_j), \mathfrak{f}_{\mathrm{S}(\mathfrak{z}_r)}(E_j))\rangle : 1 \leq j \leq t, E_j \in E\}$.
**Output**: *The line graph* $\mathcal{L}(\mathrm{H}) = (\mathrm{R}^l, \mathrm{S}^l, A)$ *of* $SNS_f H$ H, *over* $H^l = (V^l, E^l)$.

```
procedure
  Construction of underlying crisp hypergraph H^l = (V^l, E^l) of L(H)
  V^l = {V_j = E_j : ∀E_j ∈ E};
  for p := 1 to t − 1 do
    for q := p + 1 to t do
      Consider vertices V_p, V_q ∈ V^l
      if V_p ∩ V_q = ∅ then
        Edge V_p V_q belongs to E^l
      end if
    end for
  end for
  Construction of L(H)
  for r := 1 to s do
    Define R^l(z_r) over V^l = {V_j = E_j : E_j ∈ E} such that (t_{R^l(z_r)}(V_j), i_{R^l(z_r)}(V_j), f_{R^l(z_r)}(V_j)) =
    (t_{S(z_r)}(E_j), i_{S(z_r)}(E_j), f_{S(z_r)}(E_j));
    for p := 1 to t − 1 do
      for q := p + 1 to t do
        if (t_{R^l(z_r)}(V_p), i_{R^l(z_r)}(V_p), f_{R^l(z_r)}(V_p)) ≠ (0,0,0)&&(t_{R^l(z_r)}(V_q), i_{R^l(z_r)}(V_q), f_{R^l(z_r)}(V_q)) ≠
        (0,0,0) then
          SN edge between SN vertices V_p and V_q exist in L(H(z_r)) with neutrosophic grades as
          t_{S^l(z)}(V_p V_q) = t_{R^l(z)}(V_p) ∧ t_{R^l(z)}(V_q);
          i_{S^l(z)}(V_p V_q) = i_{R^l(z)}(V_p) ∧ i_{R^l(z)}(V_q);
          f_{S^l(z)}(V_p V_q) = f_{R^l(z)}(V_p) ∨ f_{R^l(z)}(V_q);
        end if
      end for
    end for
  end for
  The collection L(H) = {L(H(z_r)) : 1 ≤ r ≤ s, z_r ∈ A} is the line graph of H
end procedure
```

***Example 9*** Consider a $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ defined over $H = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_6\}$ and $E = \{v_1v_3v_5, v_2v_4v_6, v_1v_4v_5, v_2v_3v_6, v_1v_2, v_3v_4, v_5v_6\}$ given in Fig. 12.

The corresponding line graph $\mathcal{L}(\mathrm{H}) = (\mathrm{R}^l, \mathrm{S}^l, A)$ of H over $H^l = (V^l, E^l)$, where $V^l = \{V_1, V_2, V_3, V_4, V_6, V_7\} = E$ and $E^l = \{V_1V_3, V_1V_4, V_1V_5, V_1V_6, V_1V_7, V_2V_3, V_2V_4, V_2V_5, V_2V_6, V_2V_7\}$. Its graphical representation is given in Fig. 13.

**Definition 18** Let $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ be a $\mathrm{SNS}_f\mathrm{H}$ over $H = (V, E)$. The dual $\mathrm{H}^d = (\mathrm{R}^d, \mathrm{S}^d, A)$ of H is, in fact, the collection of dual $\mathrm{H}^d(\mathfrak{z})$ of $\mathrm{H}(\mathfrak{z})$, for all $\mathfrak{z}$. The dual $\mathrm{H}^d(\mathfrak{z})$ over $H^d = (V^d, E^d)$ is defined as

(1) $\mathrm{R}^d(\mathfrak{z}) = \mathrm{S}(\mathfrak{z})$ over $V^d = \{V_j = E_j : E_j \in E\}$.

(2) If $|V| = n$ then $\mathrm{S}^d(\mathfrak{z})$ is defined over $E^d = \{e_i : 1 \le i \le n\}$, where $e_i = \{V_j : v_i \in E_j, E_j \in E\}$ such that for each SN hyperedge of $\mathrm{H}^d(\mathfrak{z})$, the neutrosophic grades are

$$\mathfrak{t}_{\mathrm{S}^d(\mathfrak{z})}(e_i) = \min_{V_j \in e_i} \mathfrak{t}_{\mathrm{R}^d(\mathfrak{z})}(V_j), \quad \mathfrak{i}_{\mathrm{S}^d(\mathfrak{z})}(e_i) = \min_{V_j \in e_i} \mathfrak{i}_{\mathrm{R}^d(\mathfrak{z})}(V_j), \quad \mathfrak{f}_{\mathrm{S}^d(\mathfrak{z})}(e_i) = \max_{V_j \in e_i} \mathfrak{f}_{\mathrm{R}^d(\mathfrak{z})}(V_j).$$

Algorithm 32 illustrates the procedure for the construction of dual of a $\mathrm{SNS}_f\mathrm{H}$.

**Algorithm 32** *Dual of $SNS_fH$*

**Input**: *A $SNS_fH$ $\mathrm{H} = \{\mathrm{H}(\mathfrak{z}_r) : 1 \le r \le s, \mathfrak{z}_r \in A\}$ over $H = (V, E)$, where $\mathrm{H}(\mathfrak{z}_r) = (\mathrm{R}(\mathfrak{z}_r), \mathrm{S}(\mathfrak{z}_r))$,*
$\mathrm{R}(\mathfrak{z}_r) = \{\langle v_i, (\mathfrak{t}_{\mathrm{R}(\mathfrak{z}_r)}(v_i), \mathfrak{i}_{\mathrm{R}(\mathfrak{z}_r)}(v_i), \mathfrak{f}_{\mathrm{R}(\mathfrak{z}_r)}(v_i))\rangle : 1 \le i \le n, v_i \in V\}$ *and*
$\mathrm{S}(\mathfrak{z}_r) = \{\langle E_j, (\mathfrak{t}_{\mathrm{S}(\mathfrak{z}_r)}(E_j), \mathfrak{i}_{\mathrm{S}(\mathfrak{z}_r)}(E_j), \mathfrak{f}_{\mathrm{S}(\mathfrak{z}_r)}(E_j))\rangle : 1 \le j \le t, E_j \in E\}$.
**Output**: *The dual $\mathrm{H}^d = (\mathrm{R}^d, \mathrm{S}^d, A)$ of $SNS_fH$ $\mathrm{H}$, over $H^d = (V^d, E^d)$.*

```
procedure
  Construction of underlying crisp hypergraph H^d = (V^d, E^d) of H^d
  V^d = {V_j = E_j : ∀E_j ∈ E};
  for i := 1 to n do
    for j := 1 to t do
      if v_i ∈ E_j then
        V_j ∈ e_i;
      end if
    end for
    e_i is a hyperedge in H^d
  end for
  E^d = {e_i : 1 ≤ i ≤ n};
  Construction of H^d
  for r := 1 to s do
    Define R^d(z_r) over V^d = {V_j = E_j : E_j ∈ E} such that (t_{R^d(z_r)}(V_j), i_{R^d(z_r)}(V_j), f_{R^d(z_r)}(V_j)) =
    (t_{S(z_r)}(E_j), i_{S(z_r)}(E_j), f_{S(z_r)}(E_j));
    for i := 1 to n do
      if (t_{R^d(z_r)}(V_j), i_{R^d(z_r)}(V_j), f_{R^d(z_r)}(V_j)) ≠ (0,0,0), for all V_j ∈ e_i then
        SN hyperedge e_i exist in H^d(z_r) with neutrosophic grades as
        t_{S^d(z)}(e_i) = min_{V_j ∈ e_i} t_{R^d(z)}(V_j);
        i_{S^d(z)}(e_i) = min_{V_j ∈ e_i} i_{R^d(z)}(V_j);
        f_{S^d(z)}(e_i) = max_{V_j ∈ e_i} f_{R^d(z)}(V_j);
      end if
    end for
  end for
  The collection H^d = {H^d(z_r) : 1 ≤ r ≤ s, z_r ∈ A} is the dual of H
end procedure
```

***Example 10*** Consider a $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ defined over $H = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_6\}$ and $E = \{v_2v_4v_6, v_1v_2v_3v_5, v_1v_4, v_3v_6, v_4v_5v_6, v_1v_2, v_1v_3v_4, v_3v_5v_6, v_4v_5\}$ given in Fig. 14.

The corresponding line graph $\mathrm{H}^d = (\mathrm{R}^d, \mathrm{S}^d, A)$ of $\mathrm{H}$ over $H^d = (V^d, E^d)$, where $V^d = \{V_1, V_2, V_3, V_4, V_6, V_7, V_8, V_9\} = E$ and $E^d = \{V_1V_2, V_1V_3V_5, V_1V_4V_5, V_2V_3, V_2V_4, V_2V_5, V_1V_6, V_1V_7V_9, V_1V_8, V_6V_7, V_7V_8, V_8V_9\}$ . Its graphical representation is given in Fig. 15.

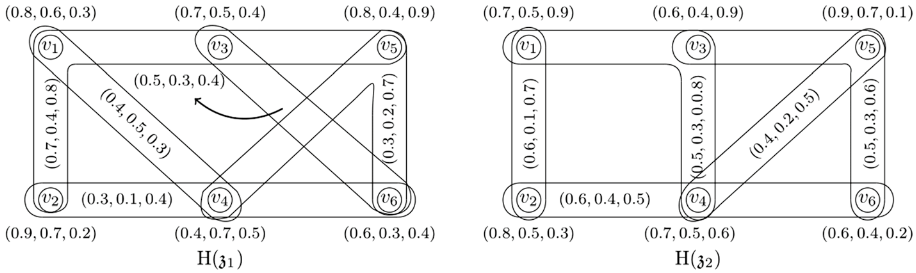



**Fig. 14** A $\text{SNS}_f$H H

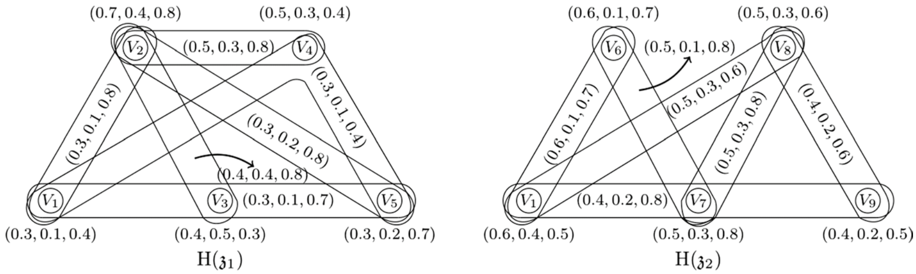



**Fig. 15** A dual $\text{SNS}_f$H $\text{H}^d$

**Definition 19** Let $\text{H} = (\text{R}, \text{S}, A)$ and $\text{H}' = (\text{R}', \text{S}', A')$ be two $\text{SNS}_f$Hs over $H = (V, E)$ and $H' = (V', E')$, respectively. Consider the SNHs $\text{H}(\mathfrak{z}) = (\text{R}(\mathfrak{z}), \text{S}(\mathfrak{z}))$ and $\text{H}'(\mathfrak{z}') = (\text{R}'(\mathfrak{z}'), \text{S}'(\mathfrak{z}'))$ of H and H′ where $\mathfrak{z} \in A$ and $\mathfrak{z}' \in A'$, respectively. Their Cartesian product is represented as $\text{H}(\mathfrak{z}) \times \text{H}'(\mathfrak{z}') = (\text{R}(\mathfrak{z}) \times \text{R}'(\mathfrak{z}'), \text{S}(\mathfrak{z}) \times \text{S}'(\mathfrak{z}'))$, where $\text{R}(\mathfrak{z}) \times \text{R}'(\mathfrak{z}')$ is a SNS over $V \times V'$ with following neutrosophic grades:

$$\text{(i)}\begin{cases} \mathfrak{t}_{\text{R}(\mathfrak{z})\times\text{R}'(\mathfrak{z}')}(v, v') = \mathfrak{t}_{\text{R}(\mathfrak{z})}(v) \wedge \mathfrak{t}_{\text{R}'(\mathfrak{z}')}(v'), \\ \mathfrak{i}_{\text{R}(\mathfrak{z})\times\text{R}'(\mathfrak{z}')}(v, v') = \mathfrak{i}_{\text{R}(\mathfrak{z})}(v) \wedge \mathfrak{i}_{\text{R}'(\mathfrak{z}')}(v'), \\ \mathfrak{f}_{\text{R}(\mathfrak{z})\times\text{R}'(\mathfrak{z}')}(v, v') = \mathfrak{f}_{\text{R}(\mathfrak{z})}(v) \vee \mathfrak{f}_{\text{R}'(\mathfrak{z}')}(v'), \end{cases}$$

for all $(v, v') \in V \times V'$, and $\text{S}(\mathfrak{z}) \times \text{S}'(\mathfrak{z}')$ is a SNS of hyperedges over $E \times E' = \{\{v\} \times E_l : v \in V, E_l \in E'\} \cup \{E_j \times \{v'\} : v' \in V', E_j \in E\}$ and the neutrosophic grades of both these types of SN hyperedges are, respectively, given below:

$$\text{(ii)}\begin{cases} \mathfrak{t}_{\text{S}(\mathfrak{z})\times\text{S}'(\mathfrak{z}')}(\{v\} \times E_l) = \mathfrak{t}_{\text{R}(\mathfrak{z})}(v) \wedge \mathfrak{t}_{\text{S}'(\mathfrak{z}')}(E_l), \\ \mathfrak{i}_{\text{S}(\mathfrak{z})\times\text{S}'(\mathfrak{z}')}(\{v\} \times E_l) = \mathfrak{i}_{\text{R}(\mathfrak{z})}(v) \wedge \mathfrak{i}_{\text{S}'(\mathfrak{z}')}(E_l), \\ \mathfrak{f}_{\text{S}(\mathfrak{z})\times\text{S}'(\mathfrak{z}')}(\{v\} \times E_l) = \mathfrak{f}_{\text{R}(\mathfrak{z})}(v) \vee \mathfrak{f}_{\text{S}'(\mathfrak{z}')}(E_l), \end{cases}$$

and

$$\text{(iii)}\begin{cases} \mathfrak{t}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(E_j \times \{v'\}) = \mathfrak{t}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'), \\ \mathfrak{i}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(E_j \times \{v'\}) = \mathfrak{i}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'), \\ \mathfrak{f}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(E_j \times \{v'\}) = \mathfrak{f}_{S(\mathfrak{z})}(E_j) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'). \end{cases}$$

As $\mathfrak{z}$ and $\mathfrak{z}'$ are arbitrary, the collection of Cartesian products $H(\mathfrak{z}) \times H'(\mathfrak{z}')$, for all $\mathfrak{z} \in A$, $\mathfrak{z}' \in A'$ is the Cartesian product $H \times H' = (R \times R', S \times S', A \times A')$ of two $SNS_f$Hs H and H′.

**Theorem 1** *The Cartesian product of two $SNS_f$Hs is a $SNS_f$H.*

***Proof*** Consider two $SNS_f$Hs $H = (R, S, A)$ and $H' = (R', S', A')$ over $H = (V, E)$ and $H' = (V', E')$, respectively. We want to show that their Cartesian product $H \times H' = (R \times R', S \times S', A \times A')$ yields a $SNS_f$H, i.e., for each $\mathfrak{z} \in A$ and $\mathfrak{z}' \in A'$, the Cartesian product of the corresponding SNHs $H(\mathfrak{z})$ and $H'(\mathfrak{z}')$ given by $H(\mathfrak{z}) \times H'(\mathfrak{z}') = (R(\mathfrak{z}) \times R'(\mathfrak{z}'), S(\mathfrak{z}) \times S'(\mathfrak{z}'))$ is also a SNH. For this, according to the definition of Cartesian product, there arise two cases:

**Case (i)** Let $v \in V$ and $E_l \in E'$ and suppose, without loss of generality that $E_l = \{v'_1, ..., v'_m\}$. Then by definition of Cartesian product

$$\begin{aligned} \mathfrak{t}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(\{v\} \times E_l) =& \mathfrak{t}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}((v, v'_1)...(v, v'_m)) \\ =& \mathfrak{t}_{R(\mathfrak{z})}(v) \wedge \mathfrak{t}_{S'(\mathfrak{z}')}(E_l) \\ \leq& \mathfrak{t}_{R(\mathfrak{z})}(v) \wedge \{\mathfrak{t}_{R'(\mathfrak{z}')}(v'_1) \wedge ... \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_m)\} \\ =& \{\mathfrak{t}_{R(\mathfrak{z})}(v) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_1)\} \wedge ... \wedge \{\mathfrak{t}_{R(\mathfrak{z})}(v) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_m)\}. \end{aligned}$$

Using $\mathfrak{t}_{R(\mathfrak{z})}(v) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v') = \mathfrak{t}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v')$, we have

$$\begin{aligned} \mathfrak{t}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(\{v\} \times E_l) \leq& \mathfrak{t}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v'_1) \wedge ... \wedge \mathfrak{t}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v'_m). \\ \mathfrak{i}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(\{v\} \times E_l) =& \mathfrak{i}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}((v, v'_1)...(v, v'_m)) \\ =& \mathfrak{i}_{R(\mathfrak{z})}(v) \wedge \mathfrak{i}_{S'(\mathfrak{z}')}(E_l) \\ \leq& \mathfrak{i}_{R(\mathfrak{z})}(v) \wedge \{\mathfrak{i}_{R'(\mathfrak{z}')}(v'_1) \wedge ... \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_m)\} \\ =& \{\mathfrak{i}_{R(\mathfrak{z})}(v) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_1)\} \wedge ... \wedge \{\mathfrak{i}_{R(\mathfrak{z})}(v) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_m)\}. \end{aligned}$$

Using $\mathfrak{i}_{R(\mathfrak{z})}(v) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v') = \mathfrak{i}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v')$, we have

$$\begin{aligned} \mathfrak{i}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(\{v\} \times E_l) \leq& \mathfrak{i}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v'_1) \wedge ... \wedge \mathfrak{i}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v'_m). \\ \mathfrak{f}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(\{v\} \times E_l) =& \mathfrak{f}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}((v, v'_1)...(v, v'_m)) \\ =& \mathfrak{f}_{R(\mathfrak{z})}(v) \vee \mathfrak{f}_{S'(\mathfrak{z}')}(E_l) \\ \leq& \mathfrak{f}_{R(\mathfrak{z})}(v) \vee \{\mathfrak{f}_{R'(\mathfrak{z}')}(v'_1) \vee ... \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_m)\} \\ =& \{\mathfrak{f}_{R(\mathfrak{z})}(v) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_1)\} \vee ... \vee \{\mathfrak{f}_{R(\mathfrak{z})}(v) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_m)\}. \end{aligned}$$

Using $\mathfrak{f}_{R(\mathfrak{z})}(v) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v') = \mathfrak{f}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v')$, we have

$$\mathfrak{f}_{S(\mathfrak{z})\times S'(\mathfrak{z}')}(\{v\} \times E_l) \leq \mathfrak{f}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v'_1) \vee ... \vee \mathfrak{f}_{R(\mathfrak{z})\times R'(\mathfrak{z}')}(v, v'_m).$$

**Case (ii)** For the case when $v' \in V'$ and $E_j \in E$, we can easily obtain

$$\mathfrak{t}_{\mathrm{S}(\mathfrak{z})\times\mathrm{S}'(\mathfrak{z}')}(E_j\times\{v'\})\leq\mathfrak{t}_{\mathrm{R}(\mathfrak{z})\times\mathrm{R}'(\mathfrak{z}')}(v_1,v)\wedge\ldots\wedge\mathfrak{t}_{\mathrm{R}(\mathfrak{z})\times\mathrm{R}'(\mathfrak{z}')}(v_m,v),$$
$$\mathfrak{i}_{\mathrm{S}(\mathfrak{z})\times\mathrm{S}'(\mathfrak{z}')}(E_j\times\{v'\})\leq\mathfrak{i}_{\mathrm{R}(\mathfrak{z})\times\mathrm{R}'(\mathfrak{z}')}(v_1,v)\wedge\ldots\wedge\mathfrak{i}_{\mathrm{R}(\mathfrak{z})\times\mathrm{R}'(\mathfrak{z}')}(v_m,v),$$
$$\mathfrak{f}_{\mathrm{S}(\mathfrak{z})\times\mathrm{S}'(\mathfrak{z}')}(E_j\times\{v'\})\leq\mathfrak{f}_{\mathrm{R}(\mathfrak{z})\times\mathrm{R}'(\mathfrak{z}')}(v_1,v)\vee\ldots\vee\mathfrak{f}_{\mathrm{R}(\mathfrak{z})\times\mathrm{R}'(\mathfrak{z}')}(v_m,v),$$

for $\mathfrak{z}\in A$, $\mathfrak{z}'\in A'$. Consequently, the Cartesian product $\mathrm{H}(\mathfrak{z})\times\mathrm{H}'(\mathfrak{z}')=(\mathrm{R}(\mathfrak{z})\times\mathrm{R}'(\mathfrak{z}'),\mathrm{S}(\mathfrak{z})\times\mathrm{S}'(\mathfrak{z}'))$ is a SNH and hence $\mathrm{H}\times\mathrm{H}'=(\mathrm{R}\times\mathrm{R}',\mathrm{S}\times\mathrm{S}',A\times A')$ is $\mathrm{SNS}_f\mathrm{H}$. □

**Definition 20** Let $\mathrm{H}=(\mathrm{R},\mathrm{S},A)$ and $\mathrm{H}'=(\mathrm{R}',\mathrm{S}',A')$ be two $\mathrm{SNS}_f$Hs over $H=(V,E)$ and $H'=(V',E')$, respectively. Consider the SNHs $\mathrm{H}(\mathfrak{z})=(\mathrm{R}(\mathfrak{z}),\mathrm{S}(\mathfrak{z}))$ and $\mathrm{H}'(\mathfrak{z}')=(\mathrm{R}'(\mathfrak{z}'),\mathrm{S}'(\mathfrak{z}'))$ of H and H′ where $\mathfrak{z}\in A$ and $\mathfrak{z}'\in A'$, respectively. Their normal product is represented as $\mathrm{H}(\mathfrak{z})\odot\mathrm{H}'(\mathfrak{z}')=(\mathrm{R}(\mathfrak{z})\odot\mathrm{R}'(\mathfrak{z}'),\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}'))$, where $\mathrm{R}(\mathfrak{z})\odot\mathrm{R}'(\mathfrak{z}')$ is a SNS over $V\times V'$ with following neutrosophic grades:

$$\text{(i)}\begin{cases}\mathfrak{t}_{\mathrm{R}(\mathfrak{z})\odot\mathrm{R}'(\mathfrak{z}')}(v,v')=\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v)\wedge\mathfrak{t}_{\mathrm{R}'(\mathfrak{z}')}(v'),\\ \mathfrak{i}_{\mathrm{R}(\mathfrak{z})\odot\mathrm{R}'(\mathfrak{z}')}(v,v')=\mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v)\wedge\mathfrak{i}_{\mathrm{R}'(\mathfrak{z}')}(v'),\\ \mathfrak{f}_{\mathrm{R}(\mathfrak{z})\odot\mathrm{R}'(\mathfrak{z}')}(v,v')=\mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v)\vee\mathfrak{f}_{\mathrm{R}'(\mathfrak{z}')}(v'),\end{cases}$$

for all $(v,v')\in V\times V'$, and $\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')$ is a SNS of hyperedges over

$$\begin{aligned}E\odot E'=&\{\{v\}\times E_l:v\in V,E_l\in E'\}\cup\{E_j\times\{v'\}:v'\in V',E_j\in E\}\cup\{(v_1,v_1')\ldots(v_r,v_r'):v_1\ldots v_r=E_j\in E,\\&v_1'\ldots v_r'\subset E_l\in E'\}\cup\{(v_1,v_1')\ldots(v_r,v_r'):v_1\ldots v_r\subset E_j\in E,v_1'\ldots v_r'=E_l\in E'\}\cup\{(v_1,v_1')\ldots(v_r,v_r'):\\&v_1\ldots v_r=E_j\in E,v_1'\ldots v_r'=E_l\in E'\},\end{aligned}$$

and the neutrosophic grades of these SN hyperedges are, respectively, given below:

$$\text{(ii)}\begin{cases}\mathfrak{t}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}(\{v\}\times E_l)=\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v)\wedge\mathfrak{t}_{\mathrm{S}'(\mathfrak{z}')}(E_l),\\ \mathfrak{i}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}(\{v\}\times E_l)=\mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v)\wedge\mathfrak{i}_{\mathrm{S}'(\mathfrak{z}')}(E_l),\\ \mathfrak{f}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}(\{v\}\times E_l)=\mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v)\vee\mathfrak{f}_{\mathrm{S}'(\mathfrak{z}')}(E_l),\end{cases}$$

$$\text{(iii)}\begin{cases}\mathfrak{t}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}(E_j\times\{v'\})=\mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j)\wedge\mathfrak{t}_{\mathrm{R}'(\mathfrak{z}')}(v'),\\ \mathfrak{i}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}(E_j\times\{v'\})=\mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j)\wedge\mathfrak{i}_{\mathrm{R}'(\mathfrak{z}')}(v'),\\ \mathfrak{f}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}(E_j\times\{v'\})=\mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(E_j)\vee\mathfrak{f}_{\mathrm{R}'(\mathfrak{z}')}(v'),\end{cases}$$

$$\text{(iv)}\begin{cases}\mathfrak{t}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}((v_1,v_1')\ldots(v_r,v_r'))=\mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j)\wedge\mathfrak{t}_{\mathrm{R}'(\mathfrak{z}')}(v_1')\wedge\ldots\wedge\mathfrak{t}_{\mathrm{R}'(\mathfrak{z}')}(v_r'),\\ \mathfrak{i}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}((v_1,v_1')\ldots(v_r,v_r'))=\mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j)\wedge\mathfrak{i}_{\mathrm{R}'(\mathfrak{z}')}(v_1')\wedge\ldots\wedge\mathfrak{i}_{\mathrm{R}'(\mathfrak{z}')}(v_r'),\\ \mathfrak{f}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}((v_1,v_1')\ldots(v_r,v_r'))=\mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(E_j)\vee\mathfrak{f}_{\mathrm{R}'(\mathfrak{z}')}(v_1')\vee\ldots\vee\mathfrak{f}_{\mathrm{R}'(\mathfrak{z}')}(v_r'),\end{cases}$$

$$\text{(v)}\begin{cases}\mathfrak{t}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}((v_1,v_1')\ldots(v_r,v_r'))=\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v_1)\wedge\ldots\wedge\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v_r)\wedge\mathfrak{t}_{\mathrm{S}'(\mathfrak{z}')}(E_l),\\ \mathfrak{i}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}((v_1,v_1')\ldots(v_r,v_r'))=\mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v_1)\wedge\ldots\wedge\mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v_r)\wedge\mathfrak{i}_{\mathrm{S}'(\mathfrak{z}')}(E_l),\\ \mathfrak{f}_{\mathrm{S}(\mathfrak{z})\odot\mathrm{S}'(\mathfrak{z}')}((v_1,v_1')\ldots(v_r,v_r'))=\mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v_1)\vee\ldots\vee\mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v_r)\vee\mathfrak{f}_{\mathrm{S}'(\mathfrak{z}')}(E_l),\end{cases}$$

and

$$\text{(vi)}\begin{cases} \mathfrak{t}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) = \mathfrak{t}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{t}_{S'(\mathfrak{z}')}(E_l), \\ \mathfrak{i}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) = \mathfrak{i}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{i}_{S'(\mathfrak{z}')}(E_l), \\ \mathfrak{f}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) = \mathfrak{f}_{S(\mathfrak{z})}(E_j) \vee \mathfrak{f}_{S'(\mathfrak{z}')}(E_l). \end{cases}$$

As $\mathfrak{z}$ and $\mathfrak{z}'$ are arbitrary, the collection of normal products $H(\mathfrak{z}) \odot H'(\mathfrak{z}')$, for all $\mathfrak{z} \in A$, $\mathfrak{z}' \in A'$ is the normal product $H \odot H' = (R \odot R', S \odot S', A \times A')$ of two $SNS_f$Hs H and H′.

**Theorem 2** *The normal product of two $SNS_f$Hs is a $SNS_f$H.*

***Proof*** Consider two $SNS_f$Hs $H = (R, S, A)$ and $H' = (R', S', A')$ over $H = (V, E)$ and $H' = (V', E')$, respectively. We want to show that their normal product $H \odot H' = (R \odot R', S \odot S', A \times AP')$ yields a $SNS_f$H, i.e., for each $\mathfrak{z} \in A$ and $\mathfrak{z}' \in A'$, the normal product of the corresponding SNHs $H(\mathfrak{z})$ and $H'(\mathfrak{z}')$ given by $H(\mathfrak{z}) \odot H'(\mathfrak{z}') = (R(\mathfrak{z}) \odot R'(\mathfrak{z}'), S(\mathfrak{z}) \odot S'(\mathfrak{z}'))$ is also a SNH. For this, according to the definition of normal product, five cases arise:

In Case (i) and Case (ii), we consider the hyperedges of the form $\{v\} \times E_l : v \in V, E_l \in E'$ or $E_j \times \{v'\} : v' \in V', E_j \in E$. The arguments similar to Theorem 1 can be employed to acquire the required results.

**Case (iii):** Consider the hyperedge $(v_1, v'_1)...(v_r, v'_r) : v_1...v_r = E_j \in E, v'_1...v'_r \subset E_l \in E'$, then by definition of normal product

$$\begin{aligned} \mathfrak{t}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) =& \mathfrak{t}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_1) \wedge ... \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_r) \\ \leq& \{\mathfrak{t}_{R(\mathfrak{z})}(v_1) \wedge ... \wedge \mathfrak{t}_{R(\mathfrak{z})}(v_r)\} \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_1) \wedge ... \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_r) \\ =& \{\mathfrak{t}_{R(\mathfrak{z})}(v_1) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_1)\} \wedge ... \wedge \{\mathfrak{t}_{R(\mathfrak{z})}(v_r) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'_r)\}. \end{aligned}$$

Using $\mathfrak{t}_{R(\mathfrak{z})}(v) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v') = \mathfrak{t}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v, v')$, we have

$$\begin{aligned} \mathfrak{t}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) \leq& \mathfrak{t}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1,v'_1) \wedge ... \wedge \mathfrak{t}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r,v'_r). \\ \mathfrak{i}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) =& \mathfrak{i}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_1) \wedge ... \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_r) \\ \leq& \{\mathfrak{i}_{R(\mathfrak{z})}(v_1) \wedge ... \wedge \mathfrak{i}_{R(\mathfrak{z})}(v_r)\} \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_1) \wedge ... \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_r) \\ =& \{\mathfrak{i}_{R(\mathfrak{z})}(v_1) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_1)\} \wedge ... \wedge \{\mathfrak{i}_{R(\mathfrak{z})}(v_r) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'_r)\}. \end{aligned}$$

Using $\mathfrak{i}_{R(\mathfrak{z})}(v) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v') = \mathfrak{i}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v, v')$, we have

$$\begin{aligned} \mathfrak{i}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) \leq& \mathfrak{i}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1,v'_1) \wedge ... \wedge \mathfrak{i}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r,v'_r). \\ \mathfrak{f}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) =& \mathfrak{f}_{S(\mathfrak{z})}(E_j) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_1) \vee ... \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_r) \\ \leq& \{\mathfrak{f}_{R(\mathfrak{z})}(v_1) \vee ... \vee \mathfrak{f}_{R(\mathfrak{z})}(v_r)\} \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_1) \vee ... \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_r) \\ =& \{\mathfrak{f}_{R(\mathfrak{z})}(v_1) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_1)\} \vee ... \vee \{\mathfrak{f}_{R(\mathfrak{z})}(v_r) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_r)\}. \end{aligned}$$

Using $\mathfrak{f}_{R(\mathfrak{z})}(v) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v') = \mathfrak{f}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v, v')$, we have

$$\mathfrak{f}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1,v'_1)...(v_r,v'_r)) \leq \mathfrak{f}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1,v'_1) \vee ... \vee \mathfrak{f}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r,v'_r).$$

**Case (iv):** Consider the hyperedge $(v_1, v'_1)...(v_r, v'_r) : v_1...v_r \subset E_j \in E, v'_1...v'_r = E_l \in E'$, we can easily obtain

$$\mathfrak{t}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) \leq \mathfrak{t}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1, v_1') \wedge ... \wedge \mathfrak{t}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r, v_r'),$$
$$\mathfrak{i}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) \leq \mathfrak{i}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1, v_1') \wedge ... \wedge \mathfrak{i}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r, v_r'),$$
$$\mathfrak{f}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) \leq \mathfrak{f}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1, v_1') \vee ... \vee \mathfrak{f}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r, v_r'),$$

**Case (v):** Consider the hyperedge $(v_1, v_1')...(v_r, v_r') : v_1...v_r = E_j \in E, v_1'...v_r' = E_l \in E'$, then by definition of normal product

$$\begin{aligned}\mathfrak{t}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) =& \mathfrak{t}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{t}_{S'(\mathfrak{z}')}(E_l)\\ \leq& \{\mathfrak{t}_{R(\mathfrak{z})}(v_1) \wedge ... \wedge \mathfrak{t}_{R(\mathfrak{z})}(v_r)\} \wedge \{\mathfrak{t}_{R'(\mathfrak{z}')}(v_1') \wedge ... \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v_r')\}\\ =& \{\mathfrak{t}_{R(\mathfrak{z})}(v_1) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v_1')\} \wedge ... \wedge \{\mathfrak{t}_{R(\mathfrak{z})}(v_r) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v_r')\}.\end{aligned}$$

Using $\mathfrak{t}_{R(\mathfrak{z})}(v) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v') = \mathfrak{t}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v, v')$, we have

$$\begin{aligned}\mathfrak{t}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) \leq& \mathfrak{t}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1, v_1') \wedge ... \wedge \mathfrak{t}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r, v_r').\\ \mathfrak{i}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) =& \mathfrak{i}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{i}_{S'(\mathfrak{z}')}(E_l)\\ \leq& \{\mathfrak{i}_{R(\mathfrak{z})}(v_1) \wedge ... \wedge \mathfrak{i}_{R(\mathfrak{z})}(v_r)\} \wedge \{\mathfrak{i}_{R'(\mathfrak{z}')}(v_1') \wedge ... \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v_r')\}\\ =& \{\mathfrak{i}_{R(\mathfrak{z})}(v_1) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v_1')\} \wedge ... \wedge \{\mathfrak{i}_{R(\mathfrak{z})}(v_r) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v_r')\}.\end{aligned}$$

Using $\mathfrak{i}_{R(\mathfrak{z})}(v) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v') = \mathfrak{i}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v, v')$, we have

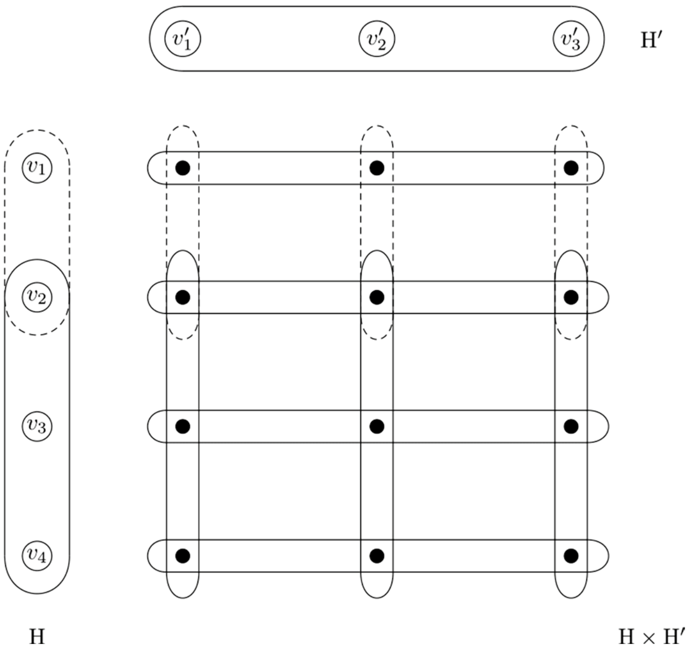



**Fig. 16** Cartesian product of H and H′

$$\mathfrak{i}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v'_1)...(v_r, v'_r)) \leq \mathfrak{i}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1, v'_1) \wedge ... \wedge \mathfrak{i}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r, v'_r).$$

$$\begin{aligned}\mathfrak{f}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v'_1)...(v_r, v'_r)) &= \mathfrak{f}_{S(\mathfrak{z})}(E_j) \vee \mathfrak{f}_{S'(\mathfrak{z}')}(E_l)\\ &\leq \{\mathfrak{f}_{R(\mathfrak{z})}(v_1) \vee ... \vee \mathfrak{f}_{R(\mathfrak{z})}(v_r)\} \vee \{\mathfrak{f}_{R'(\mathfrak{z}')}(v'_1) \vee ... \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_r)\}\\ &= \{\mathfrak{f}_{R(\mathfrak{z})}(v_1) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_1)\} \vee ... \vee \{\mathfrak{f}_{R(\mathfrak{z})}(v_r) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'_r)\}.\end{aligned}$$

Using $\mathfrak{f}_{R(\mathfrak{z})}(v) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v') = \mathfrak{f}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v, v')$, we have

$$\mathfrak{f}_{S(\mathfrak{z})\odot S'(\mathfrak{z}')}((v_1, v'_1)...(v_r, v'_r)) \leq \mathfrak{f}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_1, v'_1) \vee ... \vee \mathfrak{f}_{R(\mathfrak{z})\odot R'(\mathfrak{z}')}(v_r, v'_r).$$

for $\mathfrak{z} \in A$, $\mathfrak{z}' \in A'$. As a consequence, the normal product $H(\mathfrak{z}) \odot H'(\mathfrak{z}') = (R(\mathfrak{z}) \odot R'(\mathfrak{z}'), S(\mathfrak{z}) \odot S'(\mathfrak{z}'))$ is a SNH and hence $H \odot H' = (R \odot R', S \odot S', A \times A')$ is $SNS_fH$. □

***Example 11*** Consider $SNS_f$Hs $H = (R, S, A)$ and $H' = (R', S', A')$ on $V = \{v_1, v_2, v_3, v_4\}$ and $V' = \{v'_1, v'_2, v'_3\}$, respectively, where

$$\begin{aligned}H =& H(\mathfrak{z}) = (R(\mathfrak{z}), S(\mathfrak{z})) = (\{\langle v_1, (0.5, 0.6, 0.8)\rangle, \langle v_2, (0.8, 0.9, 0.7)\rangle, \langle v_3, (0.7, 0.3, 0.2)\rangle, \langle v_4, (0.4, 0.6, 0.9)\rangle\},\\ &\{\langle (v_1, v_2), (0.4, 0.5, 0.6)\rangle, \langle (v_2, v_3, v_4), (0.4, 0.3, 0.8)\rangle\}),\\ H' =& H'(\mathfrak{z}') = (R'(\mathfrak{z}'), S'(\mathfrak{z}')) = (\{\langle v'_1, (0.7, 0.4, 0.3)\rangle, \langle v'_2, (0.6, 0.8, 0.1)\rangle, \langle v'_3, (0.8, 0.5, 0.5)\rangle\}, \{\langle (v'_1, v'_2, v'_3),\\ &(0.6, 0.4, 0.5)\rangle\}).\end{aligned}$$

The Cartesian product $H \times H'$ of H and H′ is given by $H \times H' = H(\mathfrak{z}) \times H'(\mathfrak{z}')$, where

$$\begin{aligned}H(\mathfrak{z}) \times H'(\mathfrak{z}') =& (R(\mathfrak{z}) \times R'(\mathfrak{z}'), S(\mathfrak{z}) \times S'(\mathfrak{z}')) = (\{\langle (v_1, v'_1), (0.5, 0.4, 0.8)\rangle, \langle (v_1, v'_2), (0.5, 0.6, 0.8)\rangle, \langle (v_1, v'_3), (0.5,\\ &0.5, 0.8)\rangle, \langle (v_2, v'_1), (0.7, 0.4, 0.7)\rangle, \langle (v_2, v'_2), (0.6, 0.8, 0.7)\rangle, \langle (v_2, v'_3), (0.8, 0.5, 0.7)\rangle, \langle (v_3, v'_1), (0.7,\\ &0.3, 0.3)\rangle, \langle (v_3, v'_2), (0.6, 0.3, 0.2)\rangle, \langle (v_3, v'_3), (0.7, 0.3, 0.5)\rangle, \langle (v_4, v'_1), (0.4, 0.4, 0.9)\rangle, \langle (v_4, v'_2), (0.4,\\ &0.6, 0.9)\rangle, \langle (v_4, v'_3), (0.4, 0.5, 0.9)\rangle\}, \{\langle ((v_1, v'_1)(v_1, v'_2)(v_1, v'_3)), (0.5, 0.2, 0.6)\rangle, \langle ((v_2, v'_1)(v_2, v'_2)(v_2,\\ &v'_3)), (0.5, 0.2, 0.6)\rangle, \langle ((v_3, v'_1)(v_3, v'_2)(v_3, v'_3)), (0.5, 0.2, 0.6)\rangle, \langle ((v_4, v'_1)(v_4, v'_2)(v_4, v'_3)), (0.5, 0.2,\\ &0.6)\rangle, \langle ((v_1, v'_1)(v_2, v_1)), (0.5, 0.2, 0.6)\rangle, \langle ((v_1, v'_2)(v_2, v'_2)), (0.5, 0.2, 0.6)\rangle, \langle ((v_1, v'_3)(v_2, v'_3)), (0.4,\\ &0.3, 0.7)\rangle, \langle ((v_2, v'_1)(v_3, v'_1)(v_4, v'_1)), (0.4, 0.3, 0.7)\rangle, \langle ((v_2, v'_2)(v_3, v'_2)(v_4, v'_2)), (0.4, 0.3, 0.7)\rangle, \langle ((v_2,\\ &v'_3)(v_3, v'_3)(v_4, v'_3)), (0.4, 0.3, 0.7)\rangle\}).\end{aligned}$$

Its graphical representation is given in Fig. 16.

The normal product $H \odot H'$ of H and H′ is given by $H \odot H' = H(\mathfrak{z}) \odot H'(\mathfrak{z}')$, where

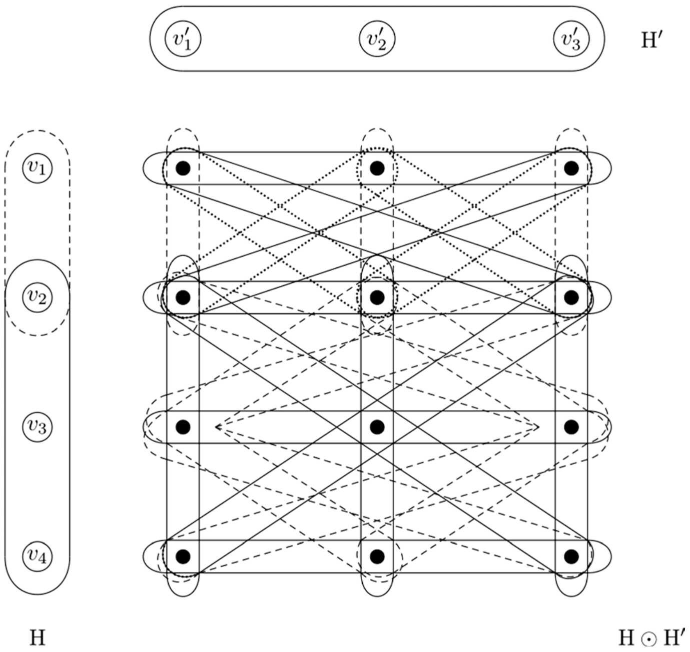



**Fig. 17** Normal product of H and H′

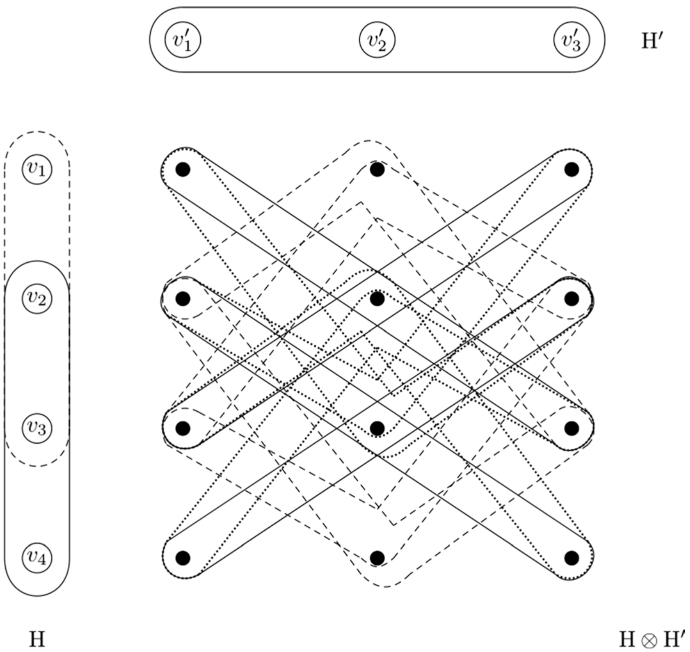



**Fig. 18** 3-Uniform direct product of H and H′

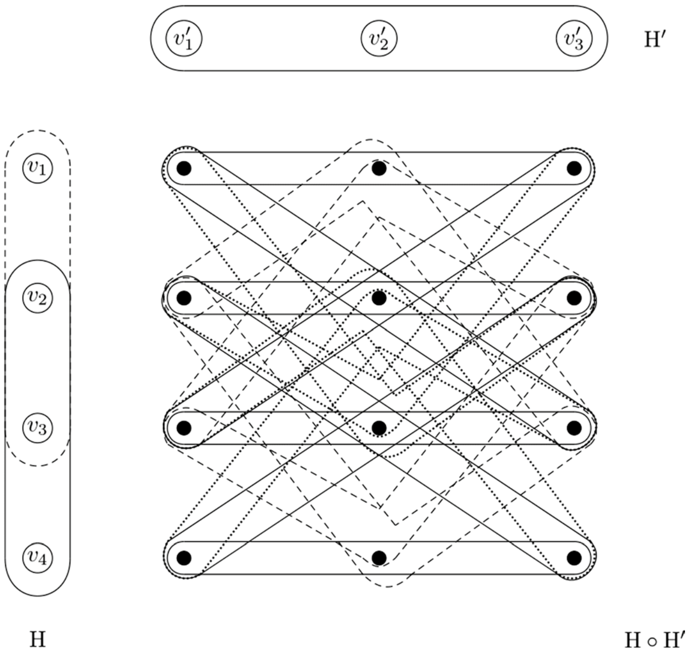



**Fig. 19** Lexicographic product of H and H′

$$\mathrm{H}(\mathfrak{z}) \odot \mathrm{H}'(\mathfrak{z}') = (\mathrm{R}(\mathfrak{z}) \odot \mathrm{R}'(\mathfrak{z}'), \mathrm{S}(\mathfrak{z}) \odot \mathrm{S}'(\mathfrak{z}')) = (\{\langle (v_1, v_1'), (0.5, 0.4, 0.8)\rangle, \langle (v_1, v_2'), (0.5, 0.6, 0.8)\rangle, \langle (v_1, v_3'), (0.5, 0.5, 0.8)\rangle, \langle (v_2, v_1'), (0.7, 0.4, 0.7)\rangle, \langle (v_2, v_2'), (0.6, 0.8, 0.7)\rangle, \langle (v_2, v_3'), (0.8, 0.5, 0.7)\rangle, \langle (v_3, v_1'), (0.7, 0.3, 0.3)\rangle, \langle (v_3, v_2'), (0.6, 0.3, 0.2)\rangle, \langle (v_3, v_3'), (0.7, 0.3, 0.5)\rangle, \langle (v_4, v_1'), (0.4, 0.4, 0.9)\rangle, \langle (v_4, v_2'), (0.4, 0.6, 0.9)\rangle, \langle (v_4, v_3'), (0.4, 0.5, 0.9)\rangle\}, \{\langle ((v_1, v_1')(v_1, v_2')(v_1, v_3')), (0.5, 0.2, 0.6)\rangle, \langle ((v_2, v_1')(v_2, v_2')(v_2, v_3')), (0.5, 0.2, 0.6)\rangle, \langle ((v_3, v_1')(v_3, v_2')(v_3, v_3')), (0.5, 0.2, 0.6)\rangle, \langle ((v_4, v_1')(v_4, v_2')(v_4, v_3')), (0.5, 0.2, 0.6)\rangle, \langle ((v_1, v_1')(v_2, v_1)), (0.5, 0.2, 0.6)\rangle, \langle ((v_1, v_2')(v_2, v_2')), (0.5, 0.2, 0.6)\rangle, \langle ((v_1, v_3')(v_2, v_3')), (0.4, 0.3, 0.7)\rangle, \langle ((v_2, v_1')(v_3, v_1')(v_4, v_1')), (0.4, 0.3, 0.7)\rangle, \langle ((v_2, v_2')(v_3, v_2')(v_4, v_2')), (0.4, 0.3, 0.7)\rangle, \langle ((v_2, v_3')(v_3, v_3')(v_4, v_3')), (0.4, 0.3, 0.7)\rangle, \langle ((v_1, v_1')(v_2, v_2')), (0.4, 0.4, 0.6)\rangle, \langle ((v_1, v_1')(v_2, v_3')), (0.4, 0.4, 0.6)\rangle, \langle ((v_1, v_2')(v_2, v_1')), (0.4, 0.4, 0.6)\rangle, \langle ((v_1, v_2')(v_2, v_3')), (0.4, 0.5, 0.6)\rangle, \langle ((v_1, v_3')(v_2, v_1')), (0.4, 0.4, 0.6)\rangle, \langle ((v_1, v_3')(v_2, v_2')), (0.4, 0.5, 0.6)\rangle, \langle ((v_2, v_1')(v_3, v_2')(v_4, v_3')), (0.4, 0.3, 0.8)\rangle, \langle ((v_2, v_1')(v_3, v_3')(v_4, v_2')), (0.4, 0.3, 0.8)\rangle, \langle ((v_2, v_2')(v_3, v_1')(v_4, v_3')), (0.4, 0.3, 0.8)\rangle, \langle ((v_2, v_2')(v_3, v_1')(v_4, v_3')), (0.4, 0.3, 0.8)\rangle, \langle ((v_2, v_2')(v_3, v_3')(v_4, v_1')), (0.4, 0.3, 0.8)\rangle, \langle ((v_2, v_3')(v_3, v_1')(v_4, v_2')), (0.4, 0.3, 0.8)\rangle, \langle ((v_2, v_3')(v_3, v_2')(v_4, v_1')), (0.4, 0.3, 0.8)\rangle\}).$$

Its graphical representation is given in Fig. 17.

## 4 *r*-Uniform single-valued neutrosophic soft hypergraphs

An $r$-uniform hypergraph is a special case of hypergraph in which each hyperedge contains exactly $r$ number of vertices in it. It is interesting to note that a graph is a 2-uniform hypergraph. The uniform hypergraphs have the ability to deal with multi-way affinity relations so we have extended this concept in $\mathrm{SNS}_f\mathrm{S}$ theory.

**Definition 21** A $\mathrm{SNS}_f\mathrm{H}$ H is said to be $r$-uniform if for all parameters $\mathfrak{z}$, the SNHs $\mathrm{H}(\mathfrak{z})$ are $r$-uniform, i.e., for each $j$, $\varepsilon_j = r$.

In order to discuss the results of $r$-uniform $\mathrm{SNS}_f\mathrm{H}$, we first define the degree and total degree of a vertex in $\mathrm{SNS}_f\mathrm{H}$.

**Definition 22** The degree $\mathfrak{d}(v)$ of a $\mathrm{SNS}_f$ vertex $v$ of a $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ is defined as the sum of degrees $\mathfrak{d}_{\mathfrak{z}}(v)$ of that vertex in all SNHs $\mathrm{H}(\mathfrak{z})$. That is,

$$\mathfrak{d}(v) = \sum_{\mathfrak{z}\in A} \mathfrak{d}_{\mathfrak{z}}(v),$$

where

$$\mathfrak{d}_{\mathfrak{z}}(v) = (\sum_{E_j \ni v} \mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j), \sum_{E_j \ni v} \mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j), \sum_{E_j \ni v} \mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(E_j)).$$

The total degree $\mathfrak{td}_{\mathfrak{z}}(v)$ of a $\mathrm{SNS}_f$ vertex $v$ of a $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ is defined as the sum of total degrees $\mathfrak{td}_{\mathfrak{z}}(v)$ of that vertex in all SNHs $\mathrm{H}(\mathfrak{z})$. That is,

$$\mathfrak{td}(v) = \sum_{\mathfrak{z}\in A} \mathfrak{td}_{\mathfrak{z}}(v),$$

where

$$\mathfrak{td}_{\mathfrak{z}}(v) = (\sum_{E_j \ni v} \mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j) + \mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v), \sum_{E_j \ni v} \mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j) + \mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v), \sum_{E_j \ni v} \mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(E_j) + \mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v)),$$

or

$$\mathfrak{td}_{\mathfrak{z}}(v) = \mathfrak{d}_{\mathfrak{z}}(v) + (\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v), \mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v), \mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v)).$$

**Theorem 3** *If* $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ *denotes the r-uniform* $\mathrm{SNS}_f\mathrm{H}$ *then degrees of its vertices satisfy the following relation*:

$$\sum_i \mathfrak{d}(v_i) = \sum_{\mathfrak{z}} (r \sum_j \mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j), r \sum_j \mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j), r \sum_j \mathfrak{f}_{\mathrm{S}(\mathfrak{z})}).$$

***Proof*** The proof to the statement is very obvious. Consider an $r$-uniform $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ over $H = (V, E)$ then $\forall j$, $\varepsilon_j = r$. Consider the SN hyperedge $E_j$ with vertices $v_1, v_2, ..., v_r$ in $\mathrm{H}(\mathfrak{z})$. Then, according to the definition of degree of vertex in $\mathrm{H}(\mathfrak{z})$, the neutrosophic grades of $E_j$ will contribute exactly once in the degree $\mathfrak{d}_{\mathfrak{z}}(v_k)$ of vertices $v_k \in E_j$, $1 \le k \le r$. Since it is true for all SN hyperedges $E_j$ of $\mathrm{H}(\mathfrak{z})$, therefore $\sum_i \mathfrak{d}_{\mathfrak{z}}(v_i) = (r \sum_j \mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j), r \sum_j \mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j), r \sum_j \mathfrak{f}_{\mathrm{S}(\mathfrak{z})})$. As $\mathfrak{z}$ is arbitrary, $\forall j, \mathfrak{z}$ the neutrosophic grades $(\mathfrak{t}_{S(\mathfrak{z})}(E_j), \mathfrak{i}_{S(\mathfrak{z})}(E_j), \mathfrak{f}_{S(\mathfrak{z})}(E_j))$ will take part exactly $r$-times in the sum of degrees of all vertices of H which generates the required result. □

**Theorem 4** *If* $H = (R, S, A)$ *denotes the r-uniform* $SNS_fH$ *then total degrees of its vertices satisfy the following relation*:

$$\sum_i \mathfrak{td}(v_i) = \sum_{\mathfrak{z}}(r\sum_j \mathfrak{t}_{S(\mathfrak{z})}(E_j(\mathfrak{z})) + \sum_i \mathfrak{t}_{R(\mathfrak{z})}(v_i), r\sum_j \mathfrak{i}_{S(\mathfrak{z})}(E_j(\mathfrak{z})) + \sum_i \mathfrak{i}_{R(\mathfrak{z})}(v_i), r\sum_j \mathfrak{f}_{S(\mathfrak{z})}(E_j(\mathfrak{z})) + \sum_i \mathfrak{f}_{R(\mathfrak{z})}(v_i)).$$

***Proof*** The proof to this theorem directly complies from the definition of total degree of vertex and Theorem 3. □

**Definition 23** Let $H = (R, S, A)$ and $H' = (R', S', A')$ be two $r$-uniform $SNS_fHs$ over $H = (V, E)$ and $H' = (V', E')$, respectively. Consider the $r$-uniform SNHs $H(\mathfrak{z}) = (R(\mathfrak{z}), S(\mathfrak{z}))$ and $H'(\mathfrak{z}') = (R'(\mathfrak{z}'), S'(\mathfrak{z}'))$ of H and H′ where $\mathfrak{z} \in A$ and $\mathfrak{z}' \in A'$, respectively. Their $r$-uniform direct product is represented as $H(\mathfrak{z}) \otimes H'(\mathfrak{z}') = (R(\mathfrak{z}) \otimes R'(\mathfrak{z}'), S(\mathfrak{z}) \otimes S'(\mathfrak{z}'))$, where $R(\mathfrak{z}) \otimes R'(\mathfrak{z}')$ is a SNS over $V \times V'$ with following neutrosophic grades:

$$\text{(i)}\begin{cases} \mathfrak{t}_{R(\mathfrak{z})\otimes R'(\mathfrak{z}')}(v, v') = \mathfrak{t}_{R(\mathfrak{z})}(v) \wedge \mathfrak{t}_{R'(\mathfrak{z}')}(v'), \\ \mathfrak{i}_{R(\mathfrak{z})\otimes R'(\mathfrak{z}')}(v, v') = \mathfrak{i}_{R(\mathfrak{z})}(v) \wedge \mathfrak{i}_{R'(\mathfrak{z}')}(v'), \\ \mathfrak{f}_{R(\mathfrak{z})\otimes R'(\mathfrak{z}')}(v, v') = \mathfrak{f}_{R(\mathfrak{z})}(v) \vee \mathfrak{f}_{R'(\mathfrak{z}')}(v'), \end{cases}$$

for all $(v, v') \in V \times V'$, and $S(\mathfrak{z}) \otimes S'(\mathfrak{z}')$ is a SNS of hyperedges over

$$E \otimes E' = \{(v_1, v_1')...(v_r, v_r') : v_1...v_r = E_j \in E, v_1'...v_r' = E_l \in E'\},$$

and the neutrosophic grades of these SN hyperedges are given below:

$$\text{(ii)}\begin{cases} \mathfrak{t}_{S(\mathfrak{z})\otimes S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) = \mathfrak{t}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{t}_{S'(\mathfrak{z}')}(E_l), \\ \mathfrak{i}_{S(\mathfrak{z})\otimes S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) = \mathfrak{i}_{S(\mathfrak{z})}(E_j) \wedge \mathfrak{i}_{S'(\mathfrak{z}')}(E_l), \\ \mathfrak{f}_{S(\mathfrak{z})\otimes S'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) = \mathfrak{f}_{S(\mathfrak{z})}(E_j) \vee \mathfrak{f}_{S'(\mathfrak{z}')}(E_l). \end{cases}$$

As $\mathfrak{z}$ and $\mathfrak{z}'$ are arbitrary, the collection of $r$-uniform direct products $H(\mathfrak{z}) \otimes H'(\mathfrak{z}')$, for all $\mathfrak{z} \in A$, $\mathfrak{z}' \in A'$ is the $r$-uniform direct product $H \otimes H' = (R \otimes R', S \otimes S', A \times A')$ of two $r$-uniform $SNS_fHs$ H and H′.

**Theorem 5** *The r-uniform direct product of two r-uniform* $SNS_fHs$ *is an r-uniform* $SNS_fH$.

***Proof*** The arguments similar to Theorem 2 [Case (v)] can be employed to get required result. □

**Definition 24** Let $H = (R, S, A)$ and $H' = (R', S', A')$ be two $r$-uniform $SNS_fHs$ over $H = (V, E)$ and $H' = (V', E')$, respectively. Consider the $r$-uniform SNHs $H(\mathfrak{z}) = (R(\mathfrak{z}), S(\mathfrak{z}))$ and $H'(\mathfrak{z}') = (R'(\mathfrak{z}'), S'(\mathfrak{z}'))$ of H and H′ where $\mathfrak{z} \in A$ and $\mathfrak{z}' \in A'$, respectively. Their lexicographic product is represented as $H(\mathfrak{z}) \circ H'(\mathfrak{z}') = (R(\mathfrak{z}) \circ R'(\mathfrak{z}'), S(\mathfrak{z}) \circ S'(\mathfrak{z}'))$, where $R(\mathfrak{z}) \circ R'(\mathfrak{z}')$ is a SNS over $V \times V'$ with following neutrosophic grades:

$$\text{(i)}\begin{cases} \mathfrak{t}_{\mathrm{R}(\mathfrak{z})\circ\mathrm{R}'(\mathfrak{z}')}(v,v') = \mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v) \wedge \mathfrak{t}_{\mathrm{R}'(\mathfrak{z}')}(v'), \\ \mathfrak{i}_{\mathrm{R}(\mathfrak{z})\circ\mathrm{R}'(\mathfrak{z}')}(v,v') = \mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v) \wedge \mathfrak{i}_{\mathrm{R}'(\mathfrak{z}')}(v'), \\ \mathfrak{f}_{\mathrm{R}(\mathfrak{z})\circ\mathrm{R}'(\mathfrak{z}')}(v,v') = \mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v) \vee \mathfrak{f}_{\mathrm{R}'(\mathfrak{z}')}(v'), \end{cases}$$

for all $(v, v') \in V \times V'$, and $\mathrm{S}(\mathfrak{z})\circ\mathrm{S}'(\mathfrak{z}')$ is a SNS of hyperedges over

$$E\circ E' = \{\{v\} \times E_l : v \in V, E_l \in E'\} \cup \{(v_1, v_1')...(v_r, v_r') : v_1...v_r = E_j \in E, v_1'...v_r' = E_l \in E'\},$$

and the neutrosophic grades of these SN hyperedges are, respectively, given below:

$$\text{(ii)}\begin{cases} \mathfrak{t}_{\mathrm{S}(\mathfrak{z})\circ\mathrm{S}'(\mathfrak{z}')}(\{v\} \times E_l) = \mathfrak{t}_{\mathrm{R}(\mathfrak{z})}(v) \wedge \mathfrak{t}_{\mathrm{S}'(\mathfrak{z}')}(E_l), \\ \mathfrak{i}_{\mathrm{S}(\mathfrak{z})\circ\mathrm{S}'(\mathfrak{z}')}(\{v\} \times E_l) = \mathfrak{i}_{\mathrm{R}(\mathfrak{z})}(v) \wedge \mathfrak{i}_{\mathrm{S}'(\mathfrak{z}')}(E_l), \\ \mathfrak{f}_{\mathrm{S}(\mathfrak{z})\circ\mathrm{S}'(\mathfrak{z}')}(\{v\} \times E_l) = \mathfrak{f}_{\mathrm{R}(\mathfrak{z})}(v) \vee \mathfrak{f}_{\mathrm{S}'(\mathfrak{z}')}(E_l), \end{cases}$$

and

$$\text{(iii)}\begin{cases} \mathfrak{t}_{\mathrm{S}(\mathfrak{z})\circ\mathrm{S}'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) = \mathfrak{t}_{\mathrm{S}(\mathfrak{z})}(E_j) \wedge \mathfrak{t}_{\mathrm{S}'(\mathfrak{z}')}(E_l), \\ \mathfrak{i}_{\mathrm{S}(\mathfrak{z})\circ\mathrm{S}'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) = \mathfrak{i}_{\mathrm{S}(\mathfrak{z})}(E_j) \wedge \mathfrak{i}_{\mathrm{S}'(\mathfrak{z}')}(E_l), \\ \mathfrak{f}_{\mathrm{S}(\mathfrak{z})\circ\mathrm{S}'(\mathfrak{z}')}((v_1, v_1')...(v_r, v_r')) = \mathfrak{f}_{\mathrm{S}(\mathfrak{z})}(E_j) \vee \mathfrak{f}_{\mathrm{S}'(\mathfrak{z}')}(E_l). \end{cases}$$

As $\mathfrak{z}$ and $\mathfrak{z}'$ are arbitrary, the collection of lexicographic products $\mathrm{H}(\mathfrak{z})\circ\mathrm{H}'(\mathfrak{z}')$, for all $\mathfrak{z} \in A$, $\mathfrak{z}' \in A'$ is the lexicographic product $\mathrm{H}\circ\mathrm{H}' = (\mathrm{R}\circ\mathrm{R}', \mathrm{S}\circ\mathrm{S}', A \times A')$ of two $r$-uniform $\mathrm{SNS}_f$Hs H and H′.

**Theorem 6** *The lexicographic product of two r-uniform $SNS_f$Hs is an r-uniform $SNS_f$H.*

***Proof*** The arguments similar to Theorem 2 [(Case (i) and Case (v)] can be employed to get required result. □

**Definition 25** Let $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ and $\mathrm{H}' = (\mathrm{R}', \mathrm{S}', A')$ be two $r$-uniform $\mathrm{SNS}_f$Hs over $H = (V, E)$ and $H' = (V', E')$, respectively. The costrong product $\mathrm{H} * \mathrm{H}'$ of H and H′ is defined as the union of lexicographic products $\mathrm{H}\circ\mathrm{H}'$ and $\mathrm{H}'\circ\mathrm{H}$, i.e., $\mathrm{H} * \mathrm{H}' = (\mathrm{H}\circ\mathrm{H}') \cup (\mathrm{H}'\circ\mathrm{H})$.

**Theorem 7** *The costrong product of two r-uniform $SNS_f$Hs is an r-uniform $SNS_f$H.*

***Example 12*** Consider two 3-uniform $\mathrm{SNS}_f$Hs $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ and $\mathrm{H}' = (\mathrm{R}', \mathrm{S}', A')$ on $V = \{v_1, v_2, v_3, v_4\}$ and $V' = \{v_1', v_2', v_3'\}$, respectively, where

$$\begin{aligned} \mathrm{H} =& \mathrm{H}(\mathfrak{z}) = (\mathrm{R}(\mathfrak{z}), \mathrm{S}(\mathfrak{z})) = (\{\langle v_1, (0.5, 0.6, 0.8)\rangle, \langle v_2, (0.8, 0.9, 0.7)\rangle, \langle v_3, (0.7, 0.3, 0.2)\rangle, \langle v_4, (0.4, 0.6, 0.9)\rangle\}, \\ & \{\langle (v_1, v_2, v_3), (0.5, 0.2, 0.6)\rangle, \langle (v_2, v_3, v_4), (0.4, 0.3, 0.7)\rangle\}), \\ \mathrm{H}' =& \mathrm{H}'(\mathfrak{z}') = (\mathrm{R}'(\mathfrak{z}'), \mathrm{S}'(\mathfrak{z}')) = (\{\langle v_1', (0.7, 0.4, 0.3)\rangle, \langle v_2', (0.6, 0.8, 0.1)\rangle, \langle v_3', (0.8, 0.5, 0.5)\rangle\}, \{\langle (v_1', v_2', v_3'), \\ & (0.6, 0.4, 0.4)\rangle\}). \end{aligned}$$

The 3-uniform direct product $\mathrm{H} \otimes \mathrm{H}'$ of H and H′ is given by $\mathrm{H} \otimes \mathrm{H}' = \mathrm{H}(\mathfrak{z}) \otimes \mathrm{H}'(\mathfrak{z}')$, where

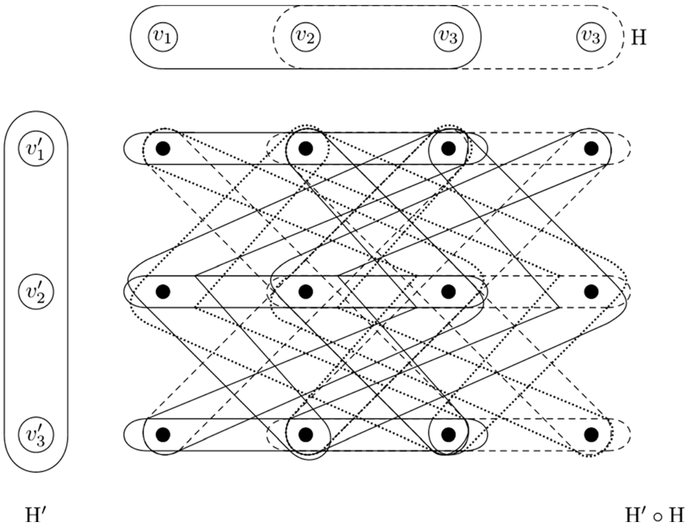



**Fig. 20** Lexicographic product of H′ and H

$$\mathrm{H}(\mathfrak{z}) \otimes \mathrm{H}'(\mathfrak{z}') = (\mathrm{R}(\mathfrak{z}) \otimes \mathrm{R}'(\mathfrak{z}'), \mathrm{S}(\mathfrak{z}) \otimes \mathrm{S}'(\mathfrak{z}')) = (\{\langle(v_1, v_1'), (0.5, 0.4, 0.8)\rangle, \langle(v_1, v_2'), (0.5, 0.6, 0.8)\rangle, \langle(v_1, v_3'), (0.5, 0.5, 0.8)\rangle, \langle(v_2, v_1'), (0.7, 0.4, 0.7)\rangle, \langle(v_2, v_2'), (0.6, 0.8, 0.7)\rangle, \langle(v_2, v_3'), (0.8, 0.5, 0.7)\rangle, \langle(v_3, v_1'), (0.7, 0.3, 0.3)\rangle, \langle(v_3, v_2'), (0.6, 0.3, 0.2)\rangle, \langle(v_3, v_3'), (0.7, 0.3, 0.5)\rangle, \langle(v_4, v_1'), (0.4, 0.4, 0.9)\rangle, \langle(v_4, v_2'), (0.4, 0.6, 0.9)\rangle, \langle(v_4, v_3'), (0.4, 0.5, 0.9)\rangle\}, \{\langle((v_1, v_1')(v_2, v_2')(v_3, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_2, v_1')(v_3, v_2')(v_1, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_3, v_1')(v_1, v_2')(v_2, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_3, v_1')(v_2, v_2')(v_1, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_2, v_1')(v_1, v_2')(v_3, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_1, v_1')(v_3, v_2')(v_2, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_2, v_1')(v_3, v_2')(v_4, v_3')), (0.4, 0.3, 0.7)\rangle, \langle((v_3, v_1')(v_4, v_2')(v_2, v_3')), (0.4, 0.3, 0.7)\rangle, \langle((v_4, v_1')(v_2, v_2')(v_3, v_3')), (0.4, 0.3, 0.7)\rangle, \langle((v_4, v_1')(v_3, v_2')(v_2, v_3')), (0.4, 0.3, 0.7)\rangle, \langle((v_3, v_1')(v_2, v_2')(v_4, v_2')), (0.4, 0.3, 0.7)\rangle, \langle((v_2, v_1')(v_4, v_2')(v_3, v_3')), (0.4, 0.3, 0.7)\rangle\}).$$

Its graphical representation is presented in Fig. 18.

The lexicographic product H∘H′ of H and H′ is given by $\mathrm{H} \circ \mathrm{H}' = \mathrm{H}(\mathfrak{z}) \circ \mathrm{H}'(\mathfrak{z}')$, where

$$\begin{aligned}\mathrm{H}(\mathfrak{z})\circ\mathrm{H}'(\mathfrak{z}') =&(\mathrm{R}(\mathfrak{z})\circ\mathrm{R}'(\mathfrak{z}'), \mathrm{S}(\mathfrak{z})\circ\mathrm{S}'(\mathfrak{z}')) = (\{\langle(v_1, v_1'), (0.5, 0.4, 0.8)\rangle, \langle(v_1, v_2'), (0.5, 0.6, 0.8)\rangle, \langle(v_1, v_3'), (0.5, \\ &0.5, 0.8)\rangle, \langle(v_2, v_1'), (0.7, 0.4, 0.7)\rangle, \langle(v_2, v_2'), (0.6, 0.8, 0.7)\rangle, \langle(v_2, v_3'), (0.8, 0.5, 0.7)\rangle, \langle(v_3, v_1'), (0.7, \\ &0.3, 0.3)\rangle, \langle(v_3, v_2'), (0.6, 0.3, 0.2)\rangle, \langle(v_3, v_3'), (0.7, 0.3, 0.5)\rangle, \langle(v_4, v_1'), (0.4, 0.4, 0.9)\rangle, \langle(v_4, v_2'), (0.4, \\ &0.6, 0.9)\rangle, \langle(v_4, v_3'), (0.4, 0.5, 0.9)\rangle\}, \{\langle((v_1, v_1')(v_2, v_2')(v_3, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_2, v_1')(v_3, v_2')(v_1, \\ &v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_3, v_1')(v_1, v_2')(v_2, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_3, v_1')(v_2, v_2')(v_1, v_3')), (0.5, 0.2, \\ &0.6)\rangle, \langle((v_2, v_1')(v_1, v_2')(v_3, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_1, v_1')(v_3, v_2')(v_2, v_3')), (0.5, 0.2, 0.6)\rangle, \langle((v_2, v_1') \\ &(v_3, v_2')(v_4, v_3')), (0.4, 0.3, 0.7)\rangle, \langle((v_3, v_1')(v_4, v_2')(v_2, v_3')), (0.4, 0.3, 0.7)\rangle, \langle((v_4, v_1')(v_2, v_2')(v_3, v_3')), \\ &(0.4, 0.3, 0.7)\rangle, \langle((v_4, v_1')(v_3, v_2')(v_2, v_3')), (0.4, 0.3, 0.7)\rangle, \langle((v_3, v_1')(v_2, v_2')(v_4, v_2')), (0.4, 0.3, 0.7)\rangle, \\ &\langle((v_2, v_1')(v_4, v_2')(v_3, v_3')), (0.4, 0.3, 0.7)\rangle, \langle((v_1, v_1')(v_1, v_2')(v_1, v_3')), (0.5, 0.4, 0.8)\rangle, \langle((v_2, v_1')(v_2, \\ &v_2')(v_2, v_3')), (0.6, 0.4, 0.7)\rangle, \langle((v_3, v_1')(v_3, v_2')(v_3, v_3')), (0.6, 0.3, 0.4)\rangle, \langle((v_4, v_1')(v_4, v_2')(v_4, v_3')), (0.4, \\ &0.4, 0.9)\rangle\}).\end{aligned}$$

Its graphical representation is given in Fig. 19.

Similarly, the lexicographic product $\mathrm{H}'\circ\mathrm{H}$ of $\mathrm{H}'$ and H is given by $\mathrm{H}'\circ\mathrm{H} = \mathrm{H}'(\mathfrak{z}')\circ\mathrm{H}(\mathfrak{z})$, where

$$\begin{aligned}\mathrm{H}'(\mathfrak{z}')\circ\mathrm{H}(\mathfrak{z}) =&(\mathrm{R}'(\mathfrak{z}')\circ\mathrm{R}(\mathfrak{z}), \mathrm{S}'(\mathfrak{z}')\circ\mathrm{S}(\mathfrak{z})) = (\{\langle(v_1', v_1), (0.5, 0.4, 0.8)\rangle, \langle(v_1', v_2), (0.7, 0.4, 0.7)\rangle, \langle(v_1', v_3), (0.7, \\ &0.3, 0.3)\rangle, \langle(v_1', v_4), (0.4, 0.4, 0.9)\rangle, \langle(v_2', v_1), (0.5, 0.6, 0.8)\rangle, \langle(v_2', v_2), (0.6, 0.8, 0.7)\rangle, \langle(v_2', v_3), (0.6, \\ &0.3, 0.2)\rangle, \langle(v_2', v_4), (0.4, 0.6, 0.9)\rangle, \langle(v_3', v_1), (0.5, 0.5, 0.8)\rangle, \langle(v_3', v_2), (0.8, 0.5, 0.7)\rangle, \langle(v_3', v_3), (0.7, \\ &0.3, 0.5)\rangle, \langle(v_3', v_4), (0.4, 0.5, 0.9)\rangle\}, \{\langle((v_1', v_1)(v_2', v_2)(v_3', v_3)), (0.5, 0.2, 0.6)\rangle, \langle((v_1', v_1)(v_2', v_3)(v_3', \\ &v_2)), (0.5, 0.2, 0.6)\rangle, \langle((v_1', v_2)(v_2', v_1)(v_3', v_3)), (0.5, 0.2, 0.6)\rangle, \langle((v_1', v_2)(v_2', v_3)(v_3', v_1)), (0.5, 0.2, \\ &0.6)\rangle, \langle((v_1', v_3)(v_2', v_1)(v_3', v_2)), (0.5, 0.2, 0.6)\rangle, \langle((v_1', v_3)(v_2', v_2)(v_3', v_1)), (0.5, 0.2, 0.6)\rangle, \langle((v_1', v_2) \\ &(v_2', v_3)(v_3', v_4)), (0.4, 0.3, 0.7)\rangle, \langle((v_1', v_2)(v_2', v_4)(v_3', v_3)), (0.4, 0.3, 0.7)\rangle, \langle((v_1', v_3)(v_2', v_2)(v_3', v_4)), \\ &(0.4, 0.3, 0.7)\rangle, \langle((v_1', v_3)(v_2', v_4)(v_3', v_2)), (0.4, 0.3, 0.7)\rangle, \langle((v_1', v_4)(v_2', v_3)(v_3', v_2)), (0.4, 0.3, 0.7)\rangle, \\ &\langle((v_1', v_4)(v_2', v_2)(v_3', v_3)), (0.4, 0.3, 0.7)\rangle, \langle((v_1', v_1)(v_1', v_2)(v_1', v_3)), (0.5, 0.2, 0.6)\rangle, \langle((v_1', v_2)(v_1', \\ &v_3)(v_1', v_4)), (0.4, 0.3, 0.7)\rangle, \langle((v_2', v_1)(v_2', v_2)(v_2', v_3)), (0.5, 0.2, 0.6)\rangle, \langle((v_2', v_2)(v_2', v_3)(v_2', v_4)), (0.4, \\ &0.4, 0.9)\rangle, \langle((v_3', v_1)(v_3', v_2)(v_3', v_3)), (0.5, 0.2, 0.6)\rangle, \langle((v_3', v_2)(v_3', v_3)(v_3', v_4)), (0.4, 0.3, 0.7)\rangle\}).\end{aligned}$$

It is graphically shown in Fig. 20.

The union of $\mathrm{SNS}_f$Hs given Figs. 19 and 20 constitutes the costrong product of H and $\mathrm{H}'$.

## 5 Regular single-valued neutrosophic soft hypergraphs

**Definition 26** A $\mathrm{SNS}_f$H $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ is said to be regular of degree $(r_1, r_2, r_3)$ if each of its SNHs $\mathrm{H}(\mathfrak{z})$ are regular of degree $(r_1, r_2, r_3)$, i.e., $\mathfrak{d}_{\mathfrak{z}}(v) = (r_1, r_2, r_3)$, $\forall v, \mathfrak{z}$.

***Example 13*** Consider the $\mathrm{SNS}_f$H $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ given in Fig. 21.

Note that the degree of each vertex $v$ in $\mathrm{H}(\mathfrak{z}_i)$ is $\mathfrak{d}_{\mathfrak{z}_i}(v) = (0.8, 0.6, 1.2)$, $i \in \{1, 2\}$. Hence, H is regular of degree (0.8, 0.6, 1.2).

**Definition 27** A $\mathrm{SNS}_f$H $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ is said to be totally regular of degree $(s_1, s_2, s_3)$ if each of its SNHs $\mathrm{H}(\mathfrak{z})$ are totally regular of degree $(s_1, s_2, s_3)$, i.e., $\mathfrak{td}_{\mathfrak{z}}(v) = (s_1, s_2, s_3)$, $\forall v, \mathfrak{z}$.

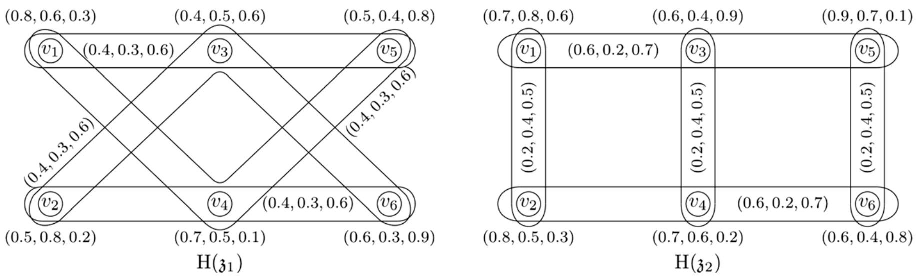



**Fig. 21** A regular $SNS_f$H H

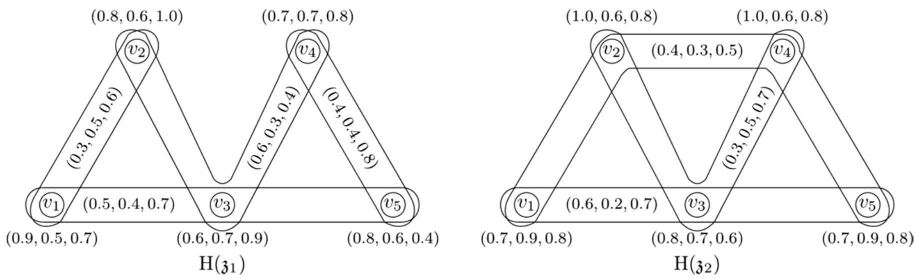



**Fig. 22** A totally regular $SNS_f$H H

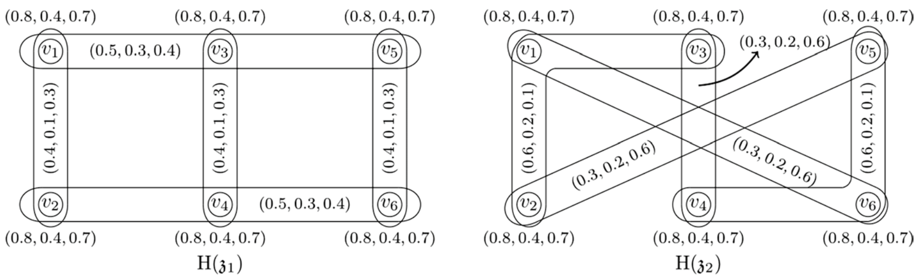



**Fig. 23** A perfectly regular $SNS_f$H H

***Example 14*** Consider the $SNS_f$H H = (R, S, $A$) shown in Fig. 22.

Note that total degree of each vertex $v$ in $H(\mathfrak{z}_i)$ is $\mathfrak{td}_{\mathfrak{z}_i}(v) = (1.7, 1.4, 2.0)$, $i \in \{1, 2\}$. Hence, H is totally regular $SNS_f$H of degree (1.7, 1.4, 2.0).

***Remark 3*** A regular $SNS_f$H may not be totally regular.

For instance, consider a regular $SNS_f$H H = (R, S, $A$) of degree (0.8, 0.6, 1.2) given in Fig. 21. Note that $\mathfrak{td}_{\mathfrak{z}_1}(v_1) = (1.6, 1.2, 1.5) \neq (1.3, 1.4, 1.4) = \mathfrak{td}_{\mathfrak{z}_1}(v_2)$. Similarly, $\mathfrak{td}_{\mathfrak{z}_2}(v_1) = (1.5, 1.4, 1.8) \neq (1.6, 1.1, 1.5) = \mathfrak{td}_{\mathfrak{z}_2}(v_2)$. Hence, H is not totally regular.

***Remark 4*** A totally regular $SNS_f$H may not be regular.

As an example, consider a totally regular SNS$_f$H H = (R, S, $A$) of degree (1.7, 1.4, 2.0) given in Fig. 22. Note that $\mathfrak{d}_{\mathfrak{z}_1}(v_1) = (0.8, 0.9, 1.3) \neq (0.9, 0.8, 2.0) = \mathfrak{d}_{\mathfrak{z}_1}(v_2)$. Similarly, $\mathfrak{d}_{\mathfrak{z}_2}(v_1) = (1.0, 0.5, 1.2) \neq (0.7, 0.8, 1.2) = \mathfrak{d}_{\mathfrak{z}_2}(v_2)$. Hence, H is not regular.

**Theorem 8** *Let* H = (R, S, $A$) *be a SNS$_f$H such that for all parameters* $\mathfrak{z}$, $\mathfrak{t}_{R(\mathfrak{z})}$, $\mathfrak{i}_{R(\mathfrak{z})}$ *and* $\mathfrak{f}_{R(\mathfrak{z})}$ *are constant functions. Then the following two statements imply one another*:

1. H *is regular SNS$_f$H.*
2. H *is totally regular SNS$_f$H.*

***Proof*** Let H = (R, S, $A$) be a SNS$_f$H and for all $v, \mathfrak{z}$, $\mathfrak{t}_{R(\mathfrak{z})}(v) = c_1$, $\mathfrak{i}_{R(\mathfrak{z})}(v) = c_2$ and $\mathfrak{f}_{R(\mathfrak{z})}(v) = c_3$, where $c_1$, $c_2$ and $c_3$ are constants from the unit closed interval. Further, suppose that H is regular of degree $(r_1, r_2, r_3)$, i.e., $\mathfrak{d}_{\mathfrak{z}}(v) = (r_1, r_2, r_3)$. Moreover, the total degree of a vertex $v$ in an arbitrary SNH H($\mathfrak{z}$) is computed as $\mathfrak{td}_{\mathfrak{z}}(v) = \mathfrak{d}_{\mathfrak{z}}(v) + (\mathfrak{t}_{R(\mathfrak{z})}(v), \mathfrak{i}_{R(\mathfrak{z})}(v), \mathfrak{f}_{R(\mathfrak{z})}(v)) = (r_1, r_2, r_3) + (c_1, c_2, c_3) = (r_1 + c_1, r_2 + c_2, r_3 + c_3), \quad \forall v$. Consequently, H is totally regular SNS$_f$H of degree $(r_1 + c_1, r_2 + c_2, r_3 + c_3)$.

For the converse part, suppose that H is totally regular SNS$_f$H of degree $(s_1, s_2, s_3)$. Then

$$
\begin{aligned}
\mathfrak{td}_{\mathfrak{z}}(v) =& (s_1, s_2, s_3) \\
\mathfrak{d}_{\mathfrak{z}}(v) + (\mathfrak{t}_{R(\mathfrak{z})}(v), \mathfrak{i}_{R(\mathfrak{z})}(v), \mathfrak{f}_{R(\mathfrak{z})}(v)) =& (s_1, s_2, s_3) \\
\mathfrak{d}_{\mathfrak{z}}(v) + (c_1, c_2, c_3) =& (s_1, s_2, s_3) \\
\mathfrak{d}_{\mathfrak{z}}(v) =& (s_1 - c_1, s_2 - c_2, s_3 - c_3)
\end{aligned}
$$

for all $v, \mathfrak{z}$. Hence, H is $(s_1 - c_1, s_2 - c_2, s_3 - c_3)$-regular and the proof ends. □

**Definition 28** A SNS$_f$H H = (R, S, $A$) is called perfectly regular if it is both regular as well as totally regular SNS$_f$H.

***Example 15*** Consider the SNS$_f$H H = (R, S, $A$) shown in Fig. 23.

Note that the degree and total degree of each vertex in H($\mathfrak{z}_i$) is $\mathfrak{d}_{\mathfrak{z}_i}(v) = (0.9, 0.4, 0.7)$ and $\mathfrak{td}_{\mathfrak{z}_i}(v) = (1.7, 0.8, 1.4)$, respectively, $i \in \{1, 2\}$. Hence, H is perfectly regular SNS$_f$H.

**Theorem 9** *If* H = (R, S, $A$) *is a perfectly regular SNS$_f$H, then for all parameters* $\mathfrak{z}$, $\mathfrak{t}_{R(\mathfrak{z})}$, $\mathfrak{i}_{R(\mathfrak{z})}$ *and* $\mathfrak{f}_{R(\mathfrak{z})}$ *are constant functions.*

***Proof*** Let H = (R, S, $A$) be a perfectly regular SNS$_f$H. This means that for all parameters $\mathfrak{z}$, the degree as well as total degree of each vertex in SNH H($\mathfrak{z}$) is same. Consequently, for all $v$, assume that $\mathfrak{d}_{\mathfrak{z}}(v) = (r_1, r_2, r_3)$ and $\mathfrak{td}_{\mathfrak{z}}(v) = (s_1, s_2, s_3)$ are the degrees and total degrees of vertices in H($\mathfrak{z}$), respectively. Using the definition of total degree of a vertex in the SNH H($\mathfrak{z}$),

$$
\begin{aligned}
\mathfrak{td}_{\mathfrak{z}}(v) =& \mathfrak{d}_{\mathfrak{z}}(v) + (\mathfrak{t}_{R(\mathfrak{z})}(v), \mathfrak{i}_{R(\mathfrak{z})}(v), \mathfrak{f}_{R(\mathfrak{z})}(v)) \\
(s_1, s_2, s_3) =& (r_1, r_2, r_3) + (\mathfrak{t}_{R(\mathfrak{z})}(v), \mathfrak{i}_{R(\mathfrak{z})}(v), \mathfrak{f}_{R(\mathfrak{z})}(v)) \\
(\mathfrak{t}_{R(\mathfrak{z})}(v), \mathfrak{i}_{R(\mathfrak{z})}(v), \mathfrak{f}_{R(\mathfrak{z})}(v)) =& (s_1 - r_1, s_2 - r_2, s_3 - r_3).
\end{aligned}
$$

Hence, $\mathfrak{z}$, $\mathfrak{t}_{R(\mathfrak{z})}$, $\mathfrak{i}_{R(\mathfrak{z})}$ and $\mathfrak{f}_{R(\mathfrak{z})}$ are constant functions. □

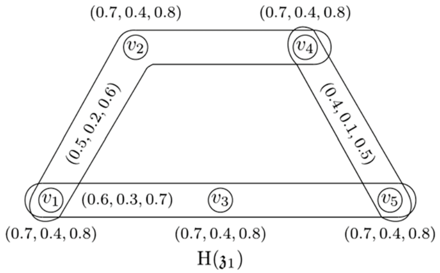



**Fig. 24** A neither regular nor totally regular $\mathrm{SNS}_f\mathrm{H}$ H

***Remark 5*** The converse of above theorem may not be true.

For instance, consider the $\mathrm{SNS}_f\mathrm{H}$ $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ given in Fig. 24. Observe that for all parameters $\mathfrak{z}$, $\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}$, $\mathfrak{i}_{\mathrm{R}(\mathfrak{z})}$ and $\mathfrak{f}_{\mathrm{R}(\mathfrak{z})}$ are constant functions. But the degree as well as total degree of vertices in $\mathrm{H}(\mathfrak{z})$ are not equal. So H is not a perfectly regular $\mathrm{SNS}_f\mathrm{H}$.

**Theorem 10** *If* $\mathrm{H} = (\mathrm{R}, \mathrm{S}, A)$ *is a regular* $\mathrm{SNS}_f\mathrm{H}$ *such that for all parameters* $\mathfrak{z}$, $\mathfrak{t}_{\mathrm{R}(\mathfrak{z})}$, $\mathfrak{i}_{\mathrm{R}(\mathfrak{z})}$ *and* $\mathfrak{f}_{\mathrm{R}(\mathfrak{z})}$ *are constant functions, then* H *is a perfectly regular* $\mathrm{SNS}_f\mathrm{H}$.

***Proof*** Straightforward. □

## 6 Single-valued neutrosophic soft directed hypergraphs

**Definition 29** Let $\overrightarrow{H} = (V, \overrightarrow{E})$ denotes a crisp directed hypergraph. A $\mathrm{SNS}_f\mathrm{DH}$ $\overrightarrow{\mathrm{H}}$ over $\overrightarrow{H}$ is denoted by the ordered triplet $\overrightarrow{\mathrm{H}} = (\mathrm{R}, \overrightarrow{\mathrm{S}}, A)$, where

(1) $(\mathrm{R}, A)$ is a $\mathrm{SNS}_f\mathrm{S}$ of vertices over $V$.
(2) $(\overrightarrow{\mathrm{S}}, A)$ is a $\mathrm{SNS}_f\mathrm{S}$ over $\overrightarrow{E}$ such that the member $E_j (1 \le j \le t)$ of $\overrightarrow{\mathrm{S}}(\mathfrak{z})$ represents the SN directed hyperedge (or SN hyperarc) in the SNDH $\overrightarrow{\mathrm{H}}(\mathfrak{z}) = (\mathrm{R}(\mathfrak{z}), \overrightarrow{\mathrm{S}}(\mathfrak{z}))$ of $\overrightarrow{\mathrm{H}}$, and its truth-membership, indeterminacy membership and falsity-membership values can be computed as

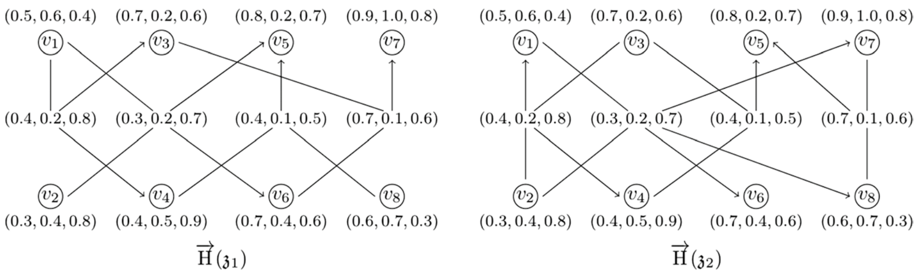



**Fig. 25** A $\mathrm{SNS}_f\mathrm{DH}$ $\overrightarrow{\mathrm{H}}$

$$\mathfrak{t}_{\vec{S}(\mathfrak{z})}(E_j) = \mathfrak{t}_{\vec{S}(\mathfrak{z})}(v_1v_2...v_m) \leq \min\{\mathfrak{t}_{R(\mathfrak{z})}(v_1), \mathfrak{t}_{R(\mathfrak{z})}(v_2), ..., \mathfrak{t}_{R(\mathfrak{z})}(v_m)\},$$
$$\mathfrak{i}_{\vec{S}(\mathfrak{z})}(E_j) = \mathfrak{i}_{\vec{S}(\mathfrak{z})}(v_1v_2...v_m) \leq \min\{\mathfrak{i}_{R(\mathfrak{z})}(v_1), \mathfrak{i}_{R(\mathfrak{z})}(v_2), ..., \mathfrak{i}_{R(\mathfrak{z})}(v_m)\},$$
$$\mathfrak{f}_{\vec{S}(\mathfrak{z})}(E_j) = \mathfrak{f}_{\vec{S}(\mathfrak{z})}(v_1v_2...v_m) \leq \max\{\mathfrak{f}_{R(\mathfrak{z})}(v_1), \mathfrak{f}_{R(\mathfrak{z})}(v_2), ..., \mathfrak{f}_{R(\mathfrak{z})}(v_m)\},$$

respectively, where $2 \leq m \leq n$.

(3) For all parameters $\mathfrak{z}$, $\bigcup_{1 \leq j \leq t} Supp(E_j) = V$, where $E_j$ denotes the SN hyperarc in $\vec{H}(\mathfrak{z})$.

Note that a SN hyperarc is denoted as an ordered pair $E_j = (t(E_j), h(E_j))$, where $t(E_j)$ and $h(E_j)$ are the disjoint sets of vertices representing the tail and head of $E_j$, respectively. All vertices of the SN hyperedge $E_j$ [either from $t(E_j)$ and $h(E_j)$] are said to be adjacent with one another.

***Example 16*** We directly present a SNS$_f$DH $\vec{H} = (R, \vec{S}, A)$ in Fig. 25.

**Definition 30** A SNS$_f$DH $\vec{H}$ is said to be a backward SNS$_f$DH if for all $\mathfrak{z}$, $\vec{H}(\mathfrak{z})$ is a backward SNDH, i.e., $\forall j$, the tail $t(E_j)$ of $E_j$ contains a single non-trivial SN vertex. In this case, each SN hyperarc $E_j$ is known as backward SN hyperarc.

**Definition 31** A SNS$_f$DH $\vec{H}$ is said to be a forward SNS$_f$DH if for all $\mathfrak{z}$, $\vec{H}(\mathfrak{z})$ is a forward SNDH, i.e., $\forall j$, the head $h(E_j)$ of $E_j$ contains a single non-trivial SN vertex. In this case, each SN hyperarc $E_j$ is known as forward SN hyperarc.

**Definition 32** A SNS$_f$DH $\vec{H}$ is said to be a backward-forward SNS$_f$DH if for all $\mathfrak{z}$, $\vec{H}(\mathfrak{z})$ is a backward-forward SNDH, i.e., $\forall j$, $E_j$ is either a backward SN hyperarc or a forward SN hyperarc.

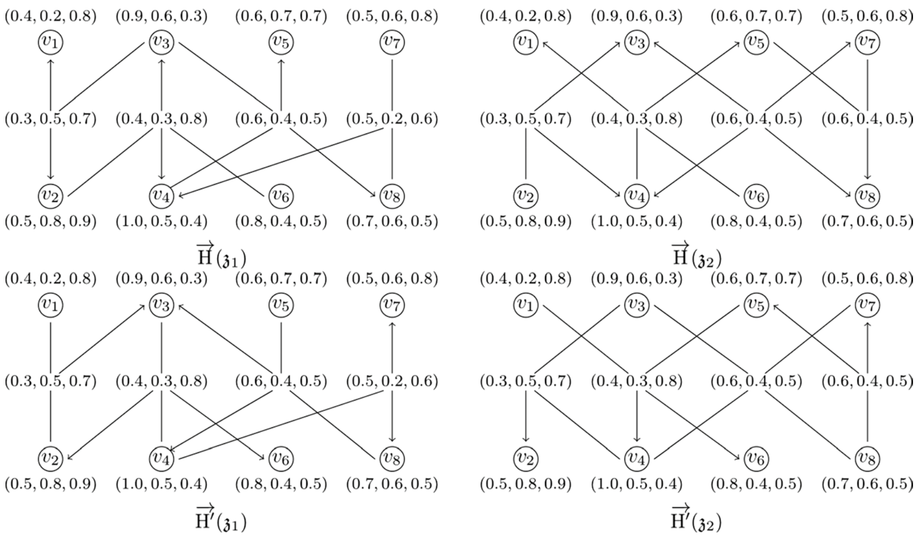



**Fig. 26** A SNS$_f$DH $\vec{H}$ and its symmetric image $\vec{H'}$

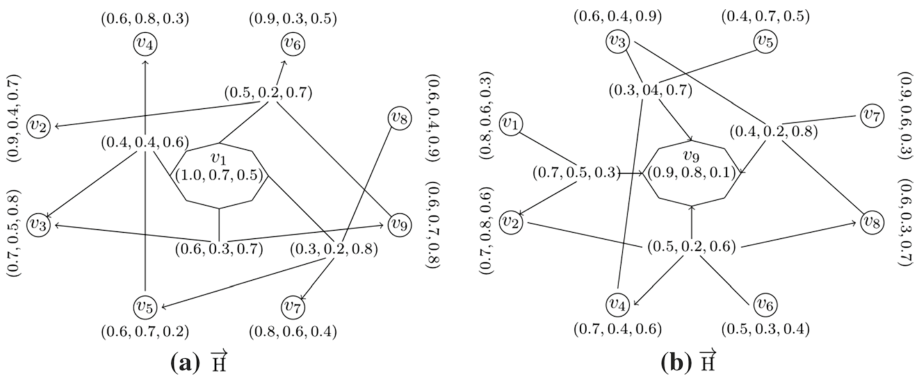



**Fig. 27** **a** A forward $\text{SNS}_f$ hyperstar. **b** A backward $\text{SNS}_f$ hyperstar

**Definition 33** Let $\overrightarrow{\mathrm{H}} = (\mathrm{R}, \overrightarrow{\mathrm{S}}, A)$ be a $\text{SNS}_f$DH over $\overrightarrow{H} = (V, \overrightarrow{E})$. The symmetric image $\overrightarrow{\mathrm{H}'} = (\mathrm{R}, \overrightarrow{\mathrm{S}'}, A)$ of $\overrightarrow{\mathrm{H}}$ is a collection of symmetric images $\overrightarrow{\mathrm{H}'}(\mathfrak{z})$ of SNDHs $\overrightarrow{\mathrm{H}}(\mathfrak{z})$, for all $\mathfrak{z}$. The symmetric image $\overrightarrow{\mathrm{H}'}(\mathfrak{z})$ has same SNS of vertices as that of $\overrightarrow{\mathrm{H}}(\mathfrak{z})$ and the SN hyperarc $E'_j = (t(E'_j), h(E'_j)) = (h(E_j), t(E_j)), \forall\, 1 \leq j \leq t$ with same neutrosophic grades as that of $E_j$. Note that the symmetric image of a backward SN hyperarc is a forward SN hyperarc and vice versa.

***Example 17*** Figure 26 shows a $\text{SNS}_f$DH $\overrightarrow{\mathrm{H}} = (\mathrm{R}, \overrightarrow{\mathrm{S}}, A)$ and its symmetric image $\overrightarrow{\mathrm{H}'} = (\mathrm{R}', \overrightarrow{\mathrm{S}'}, A)$.

**Definition 34** A $\text{SNS}_f$DH $\overrightarrow{\mathrm{H}}$ over $\overrightarrow{H} = (V, \overrightarrow{E})$ is said to be forward $\text{SNS}_f$ hyperstar of vertex $v$ if $\overrightarrow{E} = \{E_j : v \in t(E_j), \forall j\}$. Likewise, a $\text{SNS}_f$DH $\overrightarrow{\mathrm{H}}$ over $\overrightarrow{H} = (V, \overrightarrow{E})$ is said to be backward $\text{SNS}_f$ hyperstar of vertex $v$ if $\overrightarrow{E} = \{E_j : v \in h(E_j), \forall j\}$

***Example 18*** Figure 27a and b shows a forward $\text{SNS}_f$ hyperstar of vertex $v_1$ and a backward $\text{SNS}_f$ hyperstar of vertex $v_9$, respectively.

**Definition 35** Let $\overrightarrow{\mathrm{H}} = (\mathrm{R}, \overrightarrow{\mathrm{S}}, A)$ be a $\text{SNS}_f$H over $\overrightarrow{H} = (V, \overrightarrow{E})$. A SN directed hyperpath $\overrightarrow{\mathrm{P}}(\mathfrak{z})(v_1, v_p)$ from $v_1$ to $v_p$ in $\overrightarrow{\mathrm{H}}(\mathfrak{z})$ for some $\mathfrak{z} \in A$ is defined as an alternative sequence $v_1E_1v_2E_2...v_{p-1}E_{p-1}v_p$ of distinct vertices and hyperarcs such that

- $v_1 \in t(E_1)$, $v_p \in h(E_{p-1})$, $v_i \in h(E_{i-1}) \cap t(E_i); i = 2, ..., p-1$, and
- at least one of the truth-membership, indeterminacy membership and falsity-membership values is non-zero for all vertices and hyperarcs of $\overrightarrow{\mathrm{P}}(\mathfrak{z})(v_1, v_p)$.

The integer $p-1$ is called the length of $\overrightarrow{\mathrm{P}}(\mathfrak{z})(v_1, v_p)$. If $\overrightarrow{\mathrm{P}}(\mathfrak{z})(v_1, v_p)$ is a SN directed hyperpath, $\forall \mathfrak{z}$, then $v_1E_1v_2E_2...v_{p-1}E_{p-1}v_p$ is called a $\text{SNS}_f$ directed hyperpath and is denoted by $\overrightarrow{\mathrm{P}}(v_1, v_p)$. Further, If $v_1 = v_p$, then the $\text{SNS}_f$ directed hyperpath $\overrightarrow{\mathrm{P}}(v_1, v_p)$ is called $\text{SNS}_f$ directed hypercycle $\overrightarrow{\mathrm{C}}$.

## 7 $SNS_f$DHs and human nervous system

It is renowned that different brain regions and their linkages can be modeled as brain networks which efficiently illustrate the transmission of information towards and away from brain. Brain networks are not only effective in the study of brain functioning but also help in the investigation of complex brain diseases. There are different ways of construction of brain networks which will be described afterwards. Following are the grounds/basics of human nervous system.

Human nervous system controls the conscious and unconscious events of a person and conveys signals to different parts of body. Each moment, it receives a lot of information in the form of sensory signals and integrates them to decide reactions to be made by body. The nervous system is mainly composed of neurons. Neuron, the functional unit of nervous system, is a specialized cell together with its all processes. Neurons receive and conduct the sensory information through its processes/nerve fibres known as dendrites and axon, respectively, that extend out from the cell body of neuron. Nervous system has two divisions: central nervous system and peripheral nervous system. A bunch of neurons is known as nucleus and ganglion in the central and peripheral nervous system, respectively. While studying the functions of brain or in general, of nervous system, it is important to separately study the nerve fibres that carry nerve impulses *towards* and *away* from central nervous system. The nerve fibers which convey nerve impulses from central nervous system to other body parts are called efferent fibers. The well-known motor fibres are the efferent fibres which, in particular, cause contraction in skeletal muscles. The other type of nerve fibres are termed as afferent nerve fibres that transfer information from body to central nervous system. Since these fibres carry the sensational data like that of vibration, touch, temperature, pressure and pain, therefore they are also known as sensory fibres (Splitgerber 2019). Any disturbance in the pathways of nerves or nerve fibres themselves can cause serious damage. Further, the junction point of one neuron to the next neuron is called synapses. It decides that which pathway would be followed by nervous signals in the nervous system. Synapses sometimes, permit only strong signals to pass and block weak signals and at other times, choose and amplifies weak signals and transfers them in various directions.

The central nervous system refers to brain and spinal cord. Out of these, brain is one of the most important part of human body. Moreover, it is the most complex and central organ of human nervous system. It is enclosed in skull and receives the sensory information from body and its surrounding, processes that information and generates responses as well as instructs accordingly. Major divisions of brain and their subdivisions are given in Fig. 28. Conventionally, there are three major divisions of brain namely forebrain, midbrain and

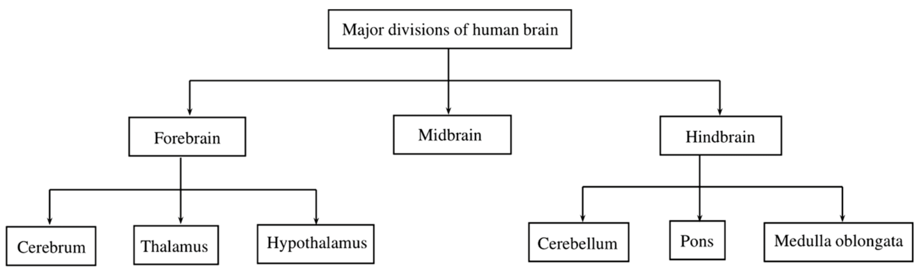
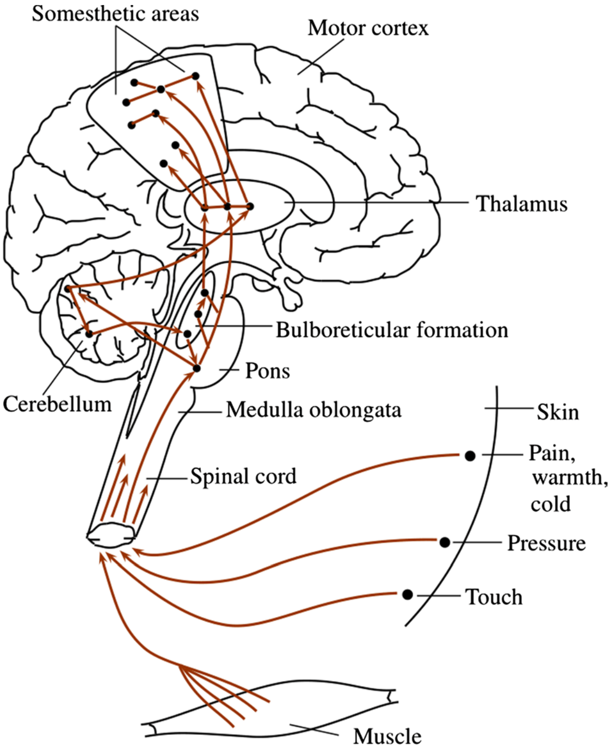



**Fig. 28** Major divisions of brain

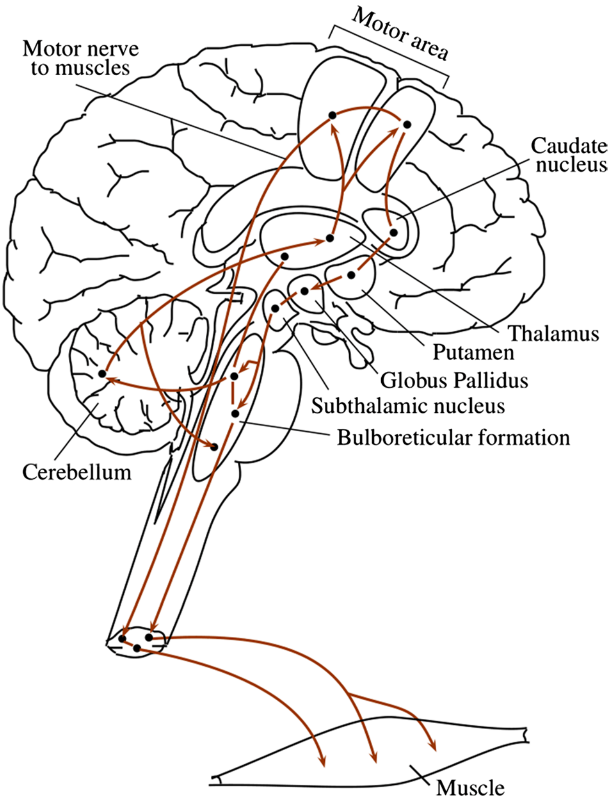
**Fig. 29** Somatosensory axis of nervous system




hindbrain. Among them, forebrain comprises cerebrum, thalamus and hypothalamus. The largest section of forebrain is the cerebrum whose different areas separately control temperature, vision and hearing, help in learning and thinking, and produce the voluntary movements of body. Putamen and globus pallidus are some of its parts. Thalamus serves as a relay station as it enables different types of information coming from body to reach in the relevant parts of cerebrum for further processing. Further, the hypothalamus provides connection between nervous system and endocrine system (hormone system) and also send signals to various glands for hormone excretion. Midbrain contains bundles of nerves heading in upward and downward direction. It is associated with the motor movement of eye muscles as well as the sleep and arousal of a person. Hind brain consists of cerebellum, pons and medulla oblongata. Cerebellum is considered as a motor structure. However, the motor movements are not commenced by cerebellum, rather the descending pathways of motor commands are modified in this part. Moreover, cerebellum also helps in speech and thinking of a person. The sensory information and motor reaction from and to the brain and facial region are terminated and originated from pons. Pons and medulla oblongata work mutually for the respiratory regulation. Medulla assists in hearing as well as equilibrium and supports movements of tongue. Spinal cord, the other main part of central nervous system, extends from brainstem in the vertebral canal. Generally, brain and spinal cord mutually work but sometimes spinal cord responds independently which is known as reflex (an automatic impulsive response to a stimulus).

The peripheral nervous system mainly comprises nerves which provide connection between central nervous system and entire body. It has two types: autonomic nervous

**Fig. 30** Skeletal motor nerve axis of nervous system




system and somatic nervous system. The autonomic nervous system supplies nerves to the involuntary body parts like glands, smooth muscles, lungs and heart to provide support in heart beat, body temperature, blood flow, emotion response and breathing. On the other hand, the somatic nervous system plays a vital role to control one's body movements. This system is accountable for almost all voluntary movements of skeletal muscles and for skin perceptiveness. It is capable to sense the orientation, location and position of body.

Generally, the brain starts activities due to the experiences of sensory organs which include eyes, ears, skin, etc. This sensory information either becomes a part of memory for late responses or causes instantaneous reactions in body. Figure 29 represents the somatic part of sensory system which transfers sensational data from skin, in particular. Subsequently, this information penetrates in different parts of central nervous system and thus controlled at distinct levels. The eventual task of nervous system is to regulate bodily activities which is acquired by the contraction of suitable skeletal muscles. Figure 30 depicts the motor responses for the skeletal muscles decided by the central nervous system. The skeletal muscles can be controlled at different levels of central nervous system which depends upon the type of sensory information received. The lower areas like spinal cord and the regions of brainstem respond to spontaneous stimuli while the higher regions, especially cerebrum considers the complex muscular movements as a result of thought processes of brain (Hall and Hall 2020).

The above discussed somatosensory and skeletal motor functionality of nervous system can be considered as the parameters of brain networks in soft set modelling. This is because mostly for better understanding, the afferent and efferent connections of different parts of nervous system are studied separately. Moreover, it is better to draw hyperarcs to show the neurophysiological signal pathway as compared to directed edges because of the occurrence of synapses of neurons at various levels of nervous system. Generally, brain regions or the nuclei are considered as nodes of network. To make these networks more realistic, the neutrosophic grades can be assigned to the nodes and hyperarcs, where the grades of nodes represent the capability, indeterminacy and incapability of brain regions to transmit information or to assist in reactions. Similarly, the neutrosophic grades of hyperarcs can be interpreted as the strength, indeterminacy and weakness of electric signal in its pathway. This will generate the $SNS_f$DH representing the activities or functioning of some part of nervous system. This criteria of nominating membership values and the construction of hyperarcs not only facilitates the study of brain functioning but also helps in the analysis of corresponding diseases of nervous system.

## 8 Conclusions

The significance of $SNS_f$S is evident from the fact that it provides information about the truthness, indeterminacy and falsity of a statement relative to each attribute of the universal set members. The discussion of hypergraphs in $SNS_f$ environment is worthwhile to express multiple linkages of real-world systems which rely on several parameters. The present study provides new graphical structures namely the $SNS_f$H as well as $SNS_f$DH. Different types of subhypergraphs have been discussed together with some operations. The line graph and dual of a $SNS_f$H have been determined with the help of algorithms. The products of the proposed structures such as the Cartesian product, normal product, direct product, lexicographic product and costrong product have been defined. Some of them are applicable to $r$-uniform $SNS_f$Hs only which can be further studied for the $SNS_f$ Hs, in general. The concept of forward and backward $SNS_f$DHs have been presented for better understanding of $SNS_f$DHs. All the operations are explained through examples. As application of the proposed model, we have briefly described the functioning of different parts of human nervous system. Further, it is illustrated that $SNS_f$DHs can be beneficial to study the activities of nervous system. The proposed model can be used to represent numerous social network as well as competing networks and in the study of various scientific and engineering applications. Our plan is to extend research in the following directions: (1) Rough single-valued neutrosophic hypergraphs (2) Regular $q$-rung picture fuzzy soft hypergraphs (3) Complex single-valued neutrosophic soft hypergraphs and (4) Single-valued neutrosophic soft competition hypergraphs.

**Acknowledgements** This project is funded by NRPU Project No. 8214, HEC Islamabad.

**Data availability:** No data were used to support this study.

## Declarations

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

## References


Akram M (2018) Single-valued neutrosophic graphs. Infosys science foundation series. Springer, Singapore. https://doi.org/10.1007/978-981-13-3522-8

Akram M, Luqman A (2017) Certain networks models using single-valued neutrosophic directed hypergraphs. J Intell Fuzzy Syst 33:575–588

Akram M, Luqman A (2020) Fuzzy hypergraphs and related extensions. Studies in fuzziness and soft computing. Springer, Singapore. https://doi.org/10.1007/978-981-15-2403-5

Akram M, Shahzadi S (2017) Neutrosophic soft graphs with application. J Intell Fuzzy Syst 32:841–858

Akram M, Shahzadi S, Saeid AB (2018) Single-valued neutrosophic hypergraphs. TWMS J Appl Eng Math 8:122–135

Atanassov KT (1986) Intuitionistic fuzzy sets: theory and applications. Fuzzy Sets Syst 20:87–96

Berge C (1973) Graphs and hypergraphs. North-Holland Publishing, Amsterdam

Berge C (1989) Hypergraphs: combinatorics of finite sets, vol 45. North-Holland Publishing, Amsterdam

Bretto A (2013) Hypergraph theory, mathematical engineering. Springer, Heidelberg

Cardoso K, Hoppen C, Trevisan V (2020) The spectrum of a class of uniform hypergraphs. Linear Algebra Appl 590:243–257

Chen C, Rajapakse I (2020) Tensor entropy for uniform hypergraphs. IEEE Trans Netw Sci Eng 7:2889–2900

Chutia R, Smarandache F (2021) Ranking of single-valued neutrosophic numbers through the index of optimism and its reasonable properties. Artif Intell Rev 55(2):1489–1518

Deli I, Broumi S (2015) Neutrosophic soft relations and some properties. Ann Fuzzy Math Inform 9:169–182

El-Hefenawy N, Metwally MA, Ahmed ZM, El-Henawy IM (2016) A review on the applications of neutrosophic sets. J Comput Theor Nanosci 13:936–944

Hall JE, Hall ME (2020) Guyton and Hall textbook of medical physiology e-Book. Elsevier, London

Hellmuth M, Ostermeier L, Stadler PF (2012) A survey on hypergraph products. Math Comput Sci 6:1–32

Karaaslan F, Davvaz B (2018) Properties of single-valued neutrosophic graphs. J Intell Fuzzy Syst 34:57–79

Liu Y, Yuan J, Duan B, Li D (2020) Quantum walks on regular uniform hypergraphs. Sci Rep 8:1–8

Mahapatra R, Samanta S, Pal M (2021) Generalized neutrosophic planar graphs and its application. J Appl Math Comput 65:693–712

Maji PK, Biswas R, Roy AR (2001a) Fuzzy soft sets. J Fuzzy Math 9:589–602

Maji PK, Biswas R, Roy AR (2001b) Intuitionistic fuzzy soft sets. J Fuzzy Math 9:677–692

Maji PK, Biswas R, Roy AR (2003) Soft set theory. Comput Math Appl 45:555–562

Maji PK (2013) Neutrosophic soft set. Ann Fuzzy Math Inform 5:157–168

Molnár B (2014) Applications of hypergraphs in informatics: a survey and opportunities for research. Ann Univ Sci Budapestinensis Sectio Comput 42:261–282

Molodtsov D (1999) Soft set theory-first results. Comput Math Appl 37:19–31

Mordeson JN, Nair PS (2001) Fuzzy graphs and fuzzy hypergraphs, 2nd edn. Physica Verlag, Heidelberg

Nguyen GN, Son LH, Ashour AS, Dey N (2019) A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. Int J Mach Learn Cybern 10:1–13

Parvathi R, Thilagavathi S, Karunambigai MG (2009) Intuitionistic fuzzy hypergraphs. Cybernet Inf Technol 9:46–53

Peng JJ, Wang JQ, Zhang HY, Chen XH (2014) An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets. Appl Soft Comput 25:336–346

Rashid MA, Ahmad S, Siddiqui MK (2020) On total uniform fuzzy soft graphs. J Intell Fuzzy Syst 39:263–275

Smarandache F, Hassan A (2016) Regular single valued neutrosophic hypergraphs. Neutrosophic Sets Syst 13:118–123

Sahin R, Liu P (2017) Correlation coefficient of single-valued neutrosophic hesitant fuzzy sets and its applications in decision making. Neural Comput Appl 28:1387–1395

Sezgin A, Atagün AO (2011) On operations of soft sets. Comput Math Appl 61:1457–1467

Shahzadi G, Akram M (2019) Hypergraphs based on Pythagorean fuzzy soft model. Math Comput Appl 24:100

Smarandache F (1998) Neutrosophy: neutrosophic probability, set and logic. American Research Press, Rehoboth

Splitgerber R (2019) Snell's clinical neuroanatomy, 8th edn. Wolters Kluwer, New York

Thilagavathi S (2018) Intuitionistic fuzzy soft hypergraph. Int J Eng Technol 7:313–315

Wang H, Smarandache F, Zhang YQ, Sunderraman R (2010) Single valued neutrosophic sets. Multispace Multistruct 4:410–413

Yang W, Cai L, Edalatpanah SA, Smarandache F (2020) Triangular single valued neutrosophic data envelopment analysis: application to hospital performance measurement. Symmetry 12:588–601
Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353
Zeng S, Shoaib M, Ali S, Smarandache F, Rashmanlou H, Mofidnakhaei F (2021) Certain properties of single-valued neutrosophic graph with application in food and agriculture organization. Int J Comput Intell Syst 14:1516–1540