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A novel logarithmic operational law and aggregation operators for trapezoidal neutrosophic number with MCGDM skill to determine most harmful virus

Tipu Sultan Haque[1] · Avishek Chakraborty[1,2] · Sankar Prasad Mondal[3] · Shariful Alam[1]

Abstract

In the current era, the theory of vagueness and multi-criteria group decision making (MCGDM) techniques are extensively applied by the researchers in disjunctive fields like recruitment policies, financial investment, design of the complex circuit, clinical diagnosis of disease, material management, etc. Recently, trapezoidal neutrosophic number (TNN) draws a major awareness to the researchers as it plays an essential role to grab the vagueness and uncertainty of daily life problems. In this article, we have focused, derived and established new logarithmic operational laws of trapezoidal neutrosophic number (TNN) where the logarithmic base μ is a positive real number. Here, logarithmic trapezoidal neutrosophic weighted arithmetic aggregation (L_{arm}) operator and logarithmic trapezoidal neutrosophic weighted geometric aggregation (L_{geo}) operator have been introduced using the logarithmic operational law. Furthermore, a new MCGDM approach is being demonstrated with the help of logarithmic operational law and aggregation operators, which has been successfully deployed to solve numerical problems. We have shown the stability and reliability of the proposed technique through sensitivity analysis. Finally, a comparative analysis has been presented to legitimize the rationality and efficiency of our proposed technique with the existing methods.

Keywords TNN · Logarithmic operational law · Aggregation operators · MCGDM

1 Introduction

Professor Zadeh [1] introduced the notion of fuzzy set theory to capture the vagueness and uncertainty of realistic problems, which was extended and expanded into intuitionistic fuzzy set(IFS) theory by Professor Attanosov [2]. To snatch the concept of uncertainty, inconsistency and indeterminacy of data in real-life problem, Professor Smarandache [3] presented the origination of neutrosophic set (NS) as an extension of IFS which contains truth membership function (μ), indeterminacy membership function (ι) and falsity membership function (σ). Recently, researchers have introduced pentagonal [4], hexagonal [5], heptagonal [6] fuzzy numbers and its application in different fields. Wang et al. manifested the conception of single-valued neutrosophic set (SVNS) [7] and interval neutrosophic set (INS) [8] which are subclasses of NSs and many other recent works [9–12] have improved and bring innovation into the NS hypothesis. Liu and Yuan [13] proposed the idea of triangular intuitionistic fuzzy set (TIFN) which is a combination of triangular fuzzy number and intuitionistic fuzzy number. Qin et al. [14] proposed a TODIM-based multi-criteria decision-making (MCDM) for TIFN. Ye [15] introduced the trapezoidal intuitionistic fuzzy number (TrIFN) and solved MCDM problem in this environment. Ye [16] manifested a novel idea of trapezoidal neutrosophic number (TNN) by mixing the concept of SVNS and trapezoidal fuzzy number and utilized it to solve an MCDM problem in trapezoidal neutrosophic (TN) arena. It is to be noted that both trapezoidal fuzzy numbers and neutrosophic numbers are important and effective tools in the field of uncertainty.

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Now, the concept of TNN can be used more fruitful in the field uncertainty to grab the impreciseness and indeterminacy in a rigorous way. In this direction, Jana et al. [17] has already defined interval traphezoidal neutrosophic numbers and apply it to solve MCGDM problem. Single valued trapezoidal neutrosophic number (SVTNN) [18] is another extension of SVNS. In SVTNN, each component is presented in the form of trapezoidal number that has truth membership degree, indeterminacy membership degree and falsity membership degree. Deli and Subas [19] manifested a ranking technique of TNN and displayed a multi-attribute decision-making (MADM) procedure. Liang et al. [20] initiated score, accuracy and certainty functions of single-valued trapezoidal neutrosophic number (SVTNN) by using center of gravity. Biswas et al. [21] defined cosine similarity measure for trapezoidal fuzzy neutrosophic numbers and presented an MADM based on it. Pramanik and Mallick [22] structured a VIKOR technique for a multi-attribute group decision making (MAGDM) in trapezoidal neutrosophic environment. Biswas et al. [23] gave the idea of TOPSIS method for MADM in TN environment, whereas Sahin et al. [24] presented some weighted arithmetic and geometric operators in SVTN environment and gave their application to MCDM problem. Abdel-Basset et al. [25] defined a type 2 neutrosophic numbers (T2NN) and manifested T2NN-TOPSIS technique to deal with a decision-making problem. Recently, Chakraborty et al. [26–29] initiated the geometrical concept of pentagonal neutrosophic number and its application in operation research, networking and graph theory arena. In this article, we have introduced new logarithmic operational laws for TNN where the logarithmic base μ is a positive real number and subsequently developed logarithmic trapezoidal neutrosophic weighted arithmetic aggregation (L_{arm}) operator and logarithmic trapezoidal neutrosophic weighted geometric aggregation (L_{geo}) operator which have been used to construct a new scheme of MCGDM process.

1.1 Motivation

In this current decade, researchers in the neutrosophic arena are mainly interested in the MCDM problems which are operators based. In the field of aggregation, the best activity is to design new operational laws. The four essential operational laws like addition, multiplication, scalar multiplication of TNN have been characterized by Ye [16]. Recently, Haque et al. [30] introduced exponential operational law where the bases are crisp numbers and the exponents are TNNs. Moreover, logarithmic operational law is a fundamental operational law in the field of aggregation. Li [31] presented logarithmic operational for IFN and developed its corresponding aggregation operators. Garg [32] set forward logarithmic operational law for

SVNS and applied it in an MADM problem. Garg [33] defined the logarithmic operational law for Pythagorean fuzzy numbers and developed corresponding aggregation operator and MCDM technique to solve the real-life problems. From the literature survey, we could not notice any logarithmic operational law for TNN till date. To mobilize this research gap, here in this research article, we have defined logarithmic operational law for TNN. Furthermore, we have successfully adopt the proposed logarithmic operator to develop new aggregation formula to aggregate several uncertain information provided by the different decision makers in an MCGDM process. Finally, we have suggested an MCGDM strategy with the help of our defined operational laws and corresponding aggregation operators namely L_{arm} and L_{geo}.

1.2 Novelties

Lots of works have been already established in the TN environment. In the meantime researchers have built different formulations and their applications in different fields of TNNs. But, there are still lots of works that can be established in this arena. In this article, we make an attempt to incorporate and address the following points:

i) To define new logarithmic operational law (LOL) for TNNs which is a useful supplement of existing operational law and analyzed their algebraic properties.

ii) To introduce new operators like logarithmic trapezoidal neutrosophic weighted arithmetic aggregation (L_{arm}) and logarithmic trapezoidal neutrosophic weighted geometric aggregation (L_{geo}) operators.

iii) Proposition of MCGDM strategy in TN environment.

iv) To demonstrate the proposed method we solved a numerical problem based on a real-life problem.

v) A sensitivity analysis is performed to show the utility and efficiency of the designed method.

1.3 Structure of the paper

The remainder of the article is organized in several sections. Section 2 presents some fundamental Definitions related with IFS and SVNS. In Section 3, we have introduced new logarithmic operational law for TNN and briefly discussed its algebraic properties. In Section 4, we have developed two aggregation operators based on our defined logarithmic operational law. In Section 5, an MCGDM method has been manifested using our defined operational laws and related aggregation operators. A numerical problem is taken to exhibit the applicability of defined logarithmic operational law and a sensitivity analysis are performed to show the utility of the designed method in Section 6. Finally, we conclude our results in Section 7.

2 Mathematical preliminaries

Basic Definitions and operations related with SVNSs and TNSs are presented as follows:

Definition 2.1 Let S be a universal set. Then

$$\widetilde{N} = \{\langle s, \ \mu(s), \ \iota(s), \ \sigma(s)\rangle; \ s \in S\}$$

is said to be single-valued neutrosophic set (SVNS) [3] on S, where $\mu : S \rightarrow [0, \ 1]$, $\iota : S \rightarrow [0, \ 1]$ and $\sigma : S \rightarrow [0, \ 1]$ with the condition $0 \leq \mu(s) + \iota(s) + \sigma(s) \leq 3$. Here, $\mu(s)$, $\iota(s)$ and $\sigma(s)$ are called the truth-membership function, indeterminacy-membership function and falsity-membership function respectively of the element to the set N. For convenience, we represent this SVNS as $\widetilde{N} = \{\langle \mu, \ \iota, \ \sigma \rangle$, where $\mu, \ \iota, \ \sigma \in [0, \ 1]$, $0 \leq \mu + \iota + \sigma \leq 3\}$ and and called as a single-valued neutrosophic number (SVNN).

Definition 2.2 Let S be a universal set. Then trapezoidal neutrosophic set $\widetilde{A}$ is defined by Ye [16] in the following form:

$$\widetilde{A} = \{\langle s, \ T(s), \ I(s), \ F(s)\rangle; \ s \in S\}$$

where $T(s) \subset [0, \ 1]$, $I(s) \subset [0, \ 1]$, $F(s) \subset [0, \ 1]$ are three trapezoidal neutrosophic numbers and $T(s) = (\alpha(s), \ \beta(s), \ \gamma(s), \ \mu(s)) : S \rightarrow [0, \ 1]$, $I(s) = (\lambda(s), \ \mu(s), \kappa(s), \ \iota(s)) : S \rightarrow [0, \ 1]$ and $F(s) = (\phi(s), \ \rho(s), \ \psi(s), \sigma(s)) : S \rightarrow [0, \ 1]$ with the condition $0 \leq \mu(s) + \iota(s) + \sigma(s) \leq 3$ for all $s \in S$. Here, $T(s)$, $I(s)$ and $F(s)$ are called the truth-membership function, indeterminacy-membership function and falsity-membership function respectively of the element to the set $\widetilde{A}$. For convenience, we represent the set as $\widetilde{A} = \{\langle(a, b, c, d), (k, l, m, n), (x, y, v, w)\rangle : 0 \leq d + n + w \leq 3\}$ and called as a trapezoidal neutrosophic number (TNN).

Proposition 2.1 *Let* $\widetilde{A}_k = \langle(a_k, b_k, c_k, d_k),$ $(l_k, m_k, n_k, p_k), \ (x_k, y_k, v_k, w_i)\rangle$ $(k = 1, 2)$ *be any two TNNs. Then, we have the following operational rules* [16]:

i) $\widetilde{A}_1 \bigoplus \widetilde{A}_2 = \langle(a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2, d_1 + d_2 - d_1 d_2),$
$(l_1 l_2, m_1 m_2, n_1 n_2, p_1 p_2),$
$(x_1 x_2, y_1 y_2, v_1 v_2, w_1 w_2)\rangle$

ii) $\widetilde{A}_1 \bigotimes \widetilde{A}_2 = \langle(a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2),$
$(l_1 + l_2 - l_1 l_2, m_1 + m_2 - m_1 m_2, n_1 + n_2 - n_1 n_2, p_1 + p_2 - p_1 p_2),$
$(x_1 + x_2 - x_1 x_2, y_1 + y_2 - y_1 y_2, v_1 + v_2 - v_1 v_2, w_1 + w_2 - w_1 w_2)\rangle$

iii) $\mu \widetilde{A}_1 = \langle(1 - (1 - a_1)^\mu, 1 - (1 - b_1)^\mu, 1 - (1 - c_1)^\mu, 1 - (1 - d_1)^\mu),$
$(l_1^\mu, m_1^\mu, n_1^\mu, k_1^\mu), (x_1^\mu, y_1^\mu, v_1^\mu, w_1^\mu)\rangle$

iv) $(\widetilde{A}_1)^\mu = \langle(a_1^\lambda, b_1^\mu, c_1^\mu, d_1^\mu), (1 - (1 - l_1)^\mu,$
$1 - (1 - m_1)^\mu, 1 - (1 - n_1)^\mu,$
$1 - (1 - k_1)^\mu),$
$(1 - (1 - x_1)^\mu, 1 - (1 - y_1)^\mu,$
$1 - (1 - v_1)^\mu, 1 - (1 - w_1)^\mu)\rangle$

Definition 2.3 Let $\widetilde{A}_s = \langle(a_s, b_s, c_s, d_s),$ $(l_s, m_s, n_s, p_s), (x_s, y_s, v_s, w_s)\rangle$ $(s = 1, 2, \cdots, p)$ be any collection of TNNs. Then the trapezoidal neutrosophic number weighted arithmetic averaging (TNNWAA) operator is defined in [16] as

$$TNNWAA(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p)$$
$$= \sum_{s=1}^{p} \phi_s \widetilde{A}_s$$
$$= \left\langle \left[1 - \prod_{s=1}^{p}(1 - a_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(1 - b_s)^{\phi_s}, \right.\right.$$
$$\left. 1 - \prod_{s=1}^{p}(1 - c_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(1 - d_s)^{\phi_s}\right],$$
$$\left[\prod_{s=1}^{p}(l_s)^{\phi_s}, \prod_{s=1}^{p}(m_s)^{\phi_s}, \prod_{s=1}^{p}(n_s)^{\phi_s}, \prod_{s=1}^{p}(p_s)^{\phi_s}\right],$$
$$\left.\left[\prod_{s=1}^{p}(x_s)^{\phi_s}, \prod_{s=1}^{p}(y_s)^{\phi_s}, \prod_{s=1}^{p}(v_s)^{\phi_s}, \prod_{s=1}^{p}(w_s)^{\phi_s}\right]\right\rangle$$

where ϕ_s $(s = 1, 2, \cdots, p)$ is the weight of $\widetilde{A}_s$ $(s = 1, 2, \cdots, p)$ with $\phi_s \in [0, 1]$ and $\sum_{s=1}^{p} \phi_s = 1$.

Definition 2.4 Let $\widetilde{A}_s = \langle(a_s, b_s, c_s, d_s),$ $(l_s, m_s, n_s, p_s), (x_s, y_s, v_s, w_s)\rangle$, $(s = 1, 2, \cdots, p)$ be collections of TNNs. Then the trapezoidal neutrosophic number weighted geometric averaging (TNNWGA) operator is defined in [16] as

$$TNNWGA(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p)$$
$$= \prod_{s=1}^{p}(\widetilde{A}_s)^{\phi_s}$$
$$= \left\langle \left[\prod_{s=1}^{p}(a_s)^{\phi_s}, \prod_{s=1}^{p}(b_s)^{\phi_s}, \prod_{s=1}^{p}(c_s)^{\phi_s}, \prod_{s=1}^{p}(d_s)^{\phi_s}\right], \right.$$
$$\left[1 - \prod_{s=1}^{p}(1 - l_s)^{\phi_s}, \ 1 - \prod_{s=1}^{p}(1 - m_s)^{\phi_s}, \right.$$
$$\left. 1 - \prod_{s=1}^{p}(1 - n_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(1 - p_s)^{\phi_s}\right],$$

$$\left[1 - \prod_{s=1}^{p}(1 - x_s)^{\phi_s}, 1 - \prod_{s=1}^{k}(1 - y_s)^{\phi_s}, \right.$$
$$\left. 1 - \prod_{s=1}^{k}(1 - v_s)^{\phi_s}, 1 - \prod_{s=1}^{k}(1 - x_s)^{\phi_s} \right] \right\rangle$$

where ϕ_s $(s = 1, 2, \cdots, p)$ is the weight of $\widetilde{A}_s$ $(s = 1, 2, \cdots, p)$ with $\phi_s \in [0, 1]$ and $\sum_{s=1}^{p}\phi_s = 1$.

2.1 Application of aggregation operators

Aggregation operators are mainly used in MCDM/MCGDM techniques to aggregate the input values of certain alternatives under the different criteria. Let, we want to evaluate an alternative under different criteria in which computational entities are in the form of TNNs. Now, we need to introduce a technique to aggregate all the evaluation values into a single value in the form of TNN. For this purpose, we have used aggregation operators TNNWAA & TNNWGA as introducded by Ye [16]. Since TNN is

an another environment in the neutrosophic field, then the above aggregation operators must have an crucial impact on MCDM/MCGDM techniques in this TN environment. Here, we have presented the following example to demonstrate the application of of above mention aggregation operators:

Example 2.1 Let someone wants to buy a new mobile phone based on the criterion of better camera quality, graphics and RAM services. Let the available alternatives are mobile companies namely X_1, X_2 and X_3, which are evaluated under the following criteria:

1) Y_1 indicates the camera quality.
2) Y_2 indicates the graphics quality services.
3) Y_3 indicates the RAM quality services.

whose weight vector is $(0.33, 0.32, 0.35)$. Figure 1 show the schematic diagram of the application of aggregation operators.

The input values of the decision making problem in TN environment are given in the following matrix

Y_1	Y_2	Y_3
$\langle(0.3, 0.7, 0.7, 0.8), (0.6, 0.7, 0.8, 0.9), (0.3, 0.4, 0.7, 0.8)\rangle$	$\langle(0.3, 0.7, 0.9, 0.9), (0.3, 0.6, 0.8, 0.9), (0.4, 0.6, 0.6, 0.7)\rangle$	$\langle(0.4, 0.5, 0.6, 0.8), (0.3, 0.5, 0.7, 0.8), (0.3, 0.5.0.6, 0.7)\rangle$
$\langle(0.5, 0.7, 0.7, 0.8), (0.7, 0.7, 0.9, 0.9), (0.1, 0.3, 0.5, 0.7)\rangle$	$\langle(0.4, 0.6, 0.8, 0.9), (0.5, 0.6, 0.7, 0.9), (0.1, 0.3, 0.5, 0.6)\rangle$	$\langle(0.6, 0.6, 0.7, 0.8), (0.6, 0.7, 0.8, 0.9), (0.1, 0.5, 0.6, 0.8)\rangle$
$\langle(0.7, 0.8, 0.8, 0.9), (0.4, 0.6, 0.8, 0.9), (0.3, 0.4, 0.6, 0.7)\rangle$	$\langle(0.7, 0.8, 0.8, 0.9), (0.2, 0.3, 0.4, 0.6), (0.4, 0.5, 0.8, 0.9)\rangle$	$\langle(0.7, 0.7, 0.9, 0.9), (0.2, 0.3, 0.4, 0.5), (0.1, 0.2, 0.3, 0.6)\rangle$

Now, if we use the operator TNNWAA on the above decision matrix, then we get the evaluation value alternatives as follows:

$$
\begin{array}{l}
X_1 \\
X_2 \\
X_3
\end{array}
\left(
\begin{array}{l}
\langle(0.3368, 0.6413, 0.7666, 0.8398), (0.3771, 0.5923, 0.7635, 0.8637), (0.3289, 0.4924, 0.6313, 0.7315)\rangle \\
\langle(0.5098, 0.6362, 0.7365, 0.8398), (0.5955, 0.6663, 0.7969, 0.9000), (0.1000, 0.3587, 0.5329, 0.6982)\rangle \\
\langle(0.7000, 0.7695, 0.8431, 0.9000), (0.2514, 0.3771, 0.5028, 0.6435), (0.2239, 0.3371, 0.5161, 0.7188)\rangle
\end{array}
\right)
$$

Again, if we use the operator TNNWGA, we get

$$
\begin{array}{l}
X_1 \\
X_2 \\
X_3
\end{array}
\left(
\begin{array}{l}
\langle(0.3318, 0.6222, 0.7188, 0.8307), (0.418, 0.6067, 0.7695, 0.8725), (0.3337, 0.5056, 0.6362, 0.7376)\rangle \\
\langle(0.4962, 0.6313, 0.7306, 0.8307), (0.6093, 0.6711, 0.8188, 0.9000), (0.1000, 0.3778, 0.5376, 0.7146)\rangle \\
\langle(0.7000, 0.7635, 0.8337, 0.9000), (0.2725, 0.4180, 0.5825, 0.7263), (0.2724, 0.3740, 0.6102, 0.7666)\rangle
\end{array}
\right)
$$

From the above example, it is observed that after utilizing the aggregation operators, we get the evaluation value of the alternatives in the aggregated form. Now, if we apply the score function [16], then we observe that, mobile company X_3 is the best option in presence of the underlying three criterion. Based on above example, we observe that if we want to evaluate some alternatives under different criteria in

TN environment, then first we need to apply the aggregation operators to convert the system into a single decision matrix. After that, utilizing the fruitful cripsification technique, we can get the associated crisp values of each alternatives. Finally, the best alternative can be determined by taking the highest crisp value among finite alternatives.

Fig. 1 Application of aggregation operators
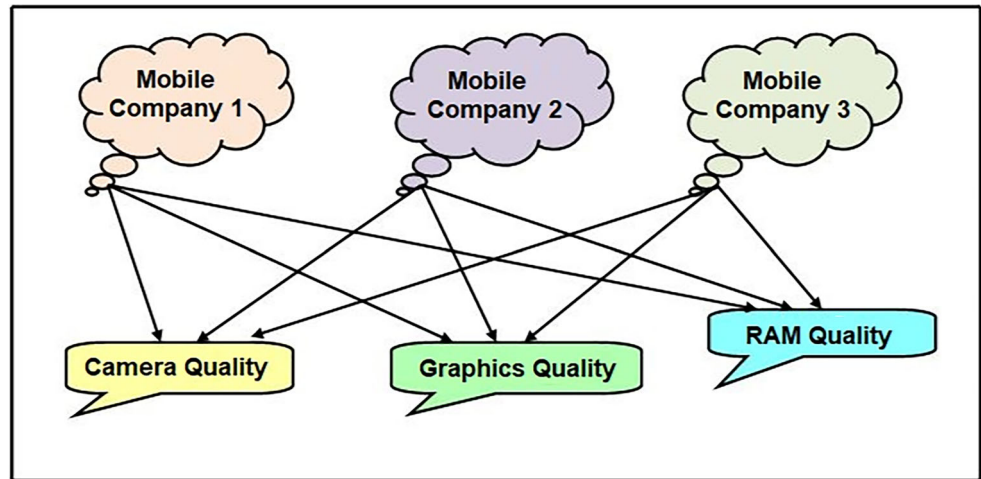
2.2 De-Neutrosophication of a TNN

De-Neutrosophication is the technique where an appreciable result is generated for crispsification. In the neutrosophic environment researchers are highly devoted to convert a TNN into a crisp number through various methods and techniques. Here, we use Removal Area Technique (RAT) to calculate de-Neutrosophication value of TNNs that is defined as follows:

Definition 2.1.1 Let $\widetilde{A} = \langle (a, b, c, d), (l, m, n, p),$
$(x, y, v, w) \rangle$ be any TNN, then the de-Neutrosophication value of $\widetilde{A}$ (utilizing Removal Area technique) is given by Chakraborty et al. [10] as

$$D_{Neu}(\widetilde{A}) = \frac{a + b + c + d + l + m + n + p + x + y + v + w}{12}.$$

Definition 2.1.2 Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be any two TNNs, then the ranking technique is defined as follows

i) If $D_{Neu}(\widetilde{A}_1) > D_{Neu}(\widetilde{A}_2)$, then $\widetilde{A}_1 > \widetilde{A}_2$
ii) If $D_{Neu}(\widetilde{A}_1) < D_{Neu}(\widetilde{A}_2)$, then $\widetilde{A}_1 < \widetilde{A}_2$.

3 Logarithmic operational law for TNN

In this section logarithmic function on TNN is defined and studied where the base (μ) is considered as positive real number. Let $\widetilde{A}$ be a TNN and $\mu > 0$ be a real number. Since in real field $\log_\mu 0$ and $\log_1 x$ are undefined, where x is a real number, so we assume that $\widetilde{A} \neq 0$, $\widetilde{A} \neq \langle [0, 0, 0, 0], [1, 1, 1, 1], [1, 1, 1, 1] \rangle$ and $\mu \neq 1$. We define the logarithm of TNN as follows:

Definition 3.1 Let V be an universal set and Let $\widetilde{A} = \langle (a, b, c, d), (l, m, n, p), (x, y, v, w) \rangle$ be any TNN. Then, we define

$$\log_\mu \widetilde{A} = \begin{cases} \begin{array}{l} \left\langle \left[1 - \log_\mu a,\, 1 - \log_\mu b,\, 1 - \log_\mu c,\, 1 - \log_\mu d\right], \left[\log_\mu(1 - l),\, \log_\mu(1 - m),\, \log_\mu(1 - n),\, \log_\mu(1 - p)\right], \right. \\ \left. \left[\log_\mu(1 - x),\, \log_\mu(1 - y),\, \log_\mu(1 - v),\, \log_\mu(1 - w)\right]\right\rangle, \\ \text{when } 0 < \mu \le \min(a, b, c, d, 1 - l, 1 - m, 1 - n, 1 - p, 1 - x, 1 - y, 1 - v, 1 - w) < 1; \\ \left\langle \left[1 - \log_{\frac{1}{\mu}} a,\, 1 - \log_{\frac{1}{\mu}} b,\, 1 - \log_{\frac{1}{\mu}} c,\, 1 - \log_{\frac{1}{\mu}} d\right], \left[\log_{\frac{1}{\mu}}(1 - l),\, \log_{\frac{1}{\mu}}(1 - m),\, \log_{\frac{1}{\mu}}(1 - n),\, \log_{\frac{1}{\mu}}(1 - p)\right] \right. \\ \left. \left[\log_{\frac{1}{\mu}}(1 - x),\, \log_{\frac{1}{\mu}}(1 - y),\, \log_{\frac{1}{\mu}}(1 - v),\, \log_{\frac{1}{\mu}}(1 - w)\right]\right\rangle, \\ \text{when } 0 < \frac{1}{\mu} \le \min(a, b, c, d, 1 - l, 1 - m, 1 - n, 1 - p, 1 - x, 1 - y, 1 - v, 1 - w) < 1. \end{array} \end{cases}$$

Here, we shall discuss some elementary Properties of $\log_\mu \widetilde{A}$ which are as follows:

Theorem 3.1 *Let* $\widetilde{A} = \langle (a, b, c, d), (l, m, n, p),$
$(x, y, v, w) \rangle$ *be a* TNN. *Then* $\log_\mu \widetilde{A}$ *is a TNN.*

Proof Let $\widetilde{A} = \langle (a, b, c, d), (l, m, n, p), (x, y, v, w) \rangle$ be a TNN. Then $a, b, c, d, l, m, n, p, x, y, v, w \in [0, 1]$ with $0 \le d + p + w \le 3$. $\square$

Case 1 When $0 < \mu \le \min (a, b, c, d, 1 - l, 1 - m, 1 - n, 1 - p) < 1$, then we have,

$$0 \le \log_\mu a, \log_\mu b, \log_\mu c, \log_\mu d, \log_\mu (1 - l), \log_\mu (1 - m), \log_\mu (1 - n), \log_\mu (1 - p), \log_\mu (1 - x), \log_\mu (1 - y), \log_\mu (1 - v), \log_\mu (1 - w) \le 1$$

Hence, $0 \le 1 - \log_\mu a, 1 - \log_\mu b, 1 - \log_\mu c, 1 - \log_\mu d, \log_\mu (1 - l), \log_\mu (1 - m), \log_\mu (1 - n), \log_\mu (1 - p), \log_\mu (1 - x), \log_\mu (1 - y), \log_\mu (1 - v), \log_\mu (1 - w) \le 1$ and $0 \le \log_\mu a + \log_\mu (1 - p) + \log_\mu (1 - w) \le 3$.

Thus, $\log_\mu \widetilde{A}$ is a TNN.

Case 2 When $0 < \frac{1}{\mu} \le \min (a, b, c, d, 1 - l, 1 - m, 1 - n, 1 - p) < 1$, then proceeding in the similar way as in the above case 1, we can prove that $\log_\mu \widetilde{A}$ is a TNN.

Thus, we conclude that $\log_\mu \widetilde{A}$ is a TNN.

Theorem 3.2 *Let* $\widetilde{A} = \langle (a, b, c, d), (l, m, n, p), (x, y, v, w) \rangle$ *be any TNN and* $0 < \mu \le \min (a, b, c, d, 1 - l, 1 - m, 1 - n, 1 - p, 1 - x, 1 - y, 1 - v, 1 - w) < 1$, *then*

i) $\mu^{\log_\mu \widetilde{A}} = \widetilde{A}$

ii) $\log_\mu \mu^{\widetilde{A}} = \widetilde{A}$

Proof i) Using the Properties 2.1 and the Definition 3.1, we get

$$\mu^{\log_\mu \widetilde{A}} = \left\langle \left[\mu^{1-(1-\log_\mu a)}, \mu^{1-(1-\log_\mu b)}, \mu^{1-(1-\log_\mu c)}, \mu^{1-(1-\log_\mu d)} \right], \left[1 - \mu^{\log_\mu (1-l)}, 1 - \mu^{\log_\mu (1-m)}, 1 - \mu^{\log_\mu (1-n)}, \right. \right.$$
$$\left. 1 - \mu^{\log_\mu (1-p)} \right], \left[1 - \mu^{\log_\mu (1-x)}, 1 - \mu^{\log_\mu (1-y)}, 1 - \mu^{\log_\mu (1-v)}, 1 - \mu^{\log_\mu (1-w)} \right] \right\rangle$$
$$= \left\langle \left[\mu^{\log_\mu a}, \mu^{\log_\mu b}, \mu^{\log_\mu c}, \mu^{\log_\mu d} \right], [1 - (1-l), 1 - (1-m), 1 - (1-n), 1 - (1-p)], [1 - (1-x), \right.$$
$$\left. 1 - (1-y), 1 - (1-v), 1 - (1-w)] \right\rangle$$
$$= \langle (a, b, c, d), (l, m, n, p), (x, y, v, w) \rangle$$
$$= \widetilde{A}.$$

ii) Again utilizing Properties 2.1 and the Definition 3.2, we get

$$\log_\mu \mu^{\widetilde{A}} = \log_\mu \left\langle [\mu^{1-a}, \mu^{1-b}, \mu^{1-c}, \mu^{1-d}], [1 - \mu^l, 1 - \mu^m, 1 - \mu^n, 1 - \mu^p], [1 - \mu^x, 1 - \mu^y, 1 - \mu^v, 1 - \mu^w] \right\rangle$$
$$= \left\langle \left[1 - \log_\mu \mu^{1-a}, 1 - \log_\mu \mu^{1-b}, 1 - \log_\mu \mu^{1-c}, 1 - \log_\mu \mu^{1-d} \right], \left[\log_\mu (1 - (1 - \mu^l)), \log_\mu (1 - (1 - \mu^m)), \right. \right.$$
$$\left. \log_\mu (1 - (1 - \mu^n)), \log_\mu (1 - (1 - \mu^p)) \right], \left[\log_\mu (1 - (1 - \mu^x)), \log_\mu (1 - (1 - \mu^y)), \log_\mu (1 - (1 - \mu^v)), \right.$$
$$\left. \log_\mu (1 - (1 - \mu^w)) \right] \right\rangle$$
$$= \langle (a, b, c, d), (l, m, n, p), (x, y, v, w) \rangle$$
$$= \widetilde{A}.$$

$\square$

Theorem 3.3 *Let* $\widetilde{A}_t = \langle (a_t, b_t, c_t, d_t), (l_t, m_t, n_t, p_t), (x_t, y_t, v_t, w_t) \rangle$ $(t = 1, 2)$ *be any two TNNs and* $0 < \mu \le \min (a_t, b_t, c_t, d_t, 1 - l_t, 1 - m_t, 1 - n_t, 1 - p_t, 1 - x_t, 1 - y_t, 1 - v_t, 1 - w_t) < 1$. *Then*

i) $\log_\mu \widetilde{A}_1 \oplus \log_\mu \widetilde{A}_2 = \log_\mu \widetilde{A}_2 \oplus \log_\mu \widetilde{A}_1$;

ii) $\log_\mu \widetilde{A}_1 \otimes \log_\mu \widetilde{A}_2 = \log_\mu A_2 \otimes \log_\mu \widetilde{A}_1$.

Proof The proof of the above Theorem follows from Properties 2.1 and Definition 3.1. $\square$

Theorem 3.4 *Let* $\widetilde{A}_t = \langle (a_t, b_t, c_t, d_t), (l_t, m_t, n_t, p_t), (x_t, y_t, v_t, w_t) \rangle$ $(t = 1, 2, 3)$ *be any three TNNs and* $0 < \mu \le \min (a_t, b_t, c_t, d_t, 1 - l_t, 1 - m_t, 1 - n_t, 1 - p_t, 1 - x_t, 1 - y_t, 1 - v_t, 1 - w_t) < 1$. *Then*

i) $\log_\mu \widetilde{A}_1 \oplus \log_\mu \widetilde{A}_2 \oplus \log_\mu \widetilde{A}_3 = \log_\mu \widetilde{A}_3 \oplus \log_\mu \widetilde{A}_2 \oplus \log_\mu A_1$;

ii) $\log_\mu \widetilde{A}_1 \otimes \log_\mu \widetilde{A}_2 \otimes \log_\mu \widetilde{A}_3 = \log_\mu \widetilde{A}_3 \otimes \log_\mu \widetilde{A}_2 \otimes \log_\mu \widetilde{A}_1$.

Proof The proof of the above Theorem follows from Properties 2.1 and Definition 3.1. $\qquad\square$

Theorem 3.5 *Let* $\widetilde{A}_t = \langle(a_t, b_t, c_t, d_t), (l_t, m_t, n_t, p_t), (x_t, y_t, v_t, w_t)\rangle$ $(t = 1, 2)$ *be any two TNNs and* $0 < \mu \le$ *min* $(a_t, b_t, c_t, d_t, 1 - l_t, 1 - m_t, 1 - n_t, 1 - p_t, 1 - x_t, 1 - y_t, 1 - v_t, 1 - w_t) < 1$. *Then*

i) $k(\log_\mu \widetilde{A}_1 \oplus \log_\mu \widetilde{A}_2) = k \log_\mu \widetilde{A}_1 \oplus k \log_\mu \widetilde{A}_2;$

ii) $(\log_\mu \widetilde{A}_1 \otimes \log_\mu \widetilde{A}_2)^k = (\log_\mu \widetilde{A}_1)^k \otimes (\log_\mu \widetilde{A}_2)^k;$

iii) $k_1 \log_\mu \widetilde{A}_1 \oplus k_2 \log_\mu \widetilde{A}_1 = (k_1 + k_2) \log_\mu \widetilde{A}_1;$

iv) $(\log_\mu \widetilde{A}_1)^{k_1} \otimes (\log_\mu \widetilde{A}_1)^{k_2} = (\log_\mu \widetilde{A}_1)^{k_1+k_2};$

v) $((\log_\mu \widetilde{A}_1)^{k_1})^{k_2} = (\log_\mu \widetilde{A}_1)^{k_1 k_2}$, *where* $k, k_1,$ & k_2 *are positive real numbers.*

Proof i) We know,

$$\log_\mu \widetilde{A}_1$$
$$= \langle[1 - \log_\mu a_1, 1 - \log_\mu b_1, 1 - \log_\mu c_1, 1 - \log_\mu d_1], [\log_\mu(1 - l_1), \log_\mu(1 - m_1), \log_\mu(1 - n_1), \log_\mu(1 - p_1)],$$
$$[\log_\mu(1 - x_1), \log_\mu(1 - y_1), \log_\mu(1 - v_1), \log_\mu(1 - w_1)]\rangle.$$

$$\log_\mu \widetilde{A}_2$$
$$= \langle[1 - \log_\mu a_2, 1 - \log_\mu b_2, 1 - \log_\mu c_2, 1 - \log_\mu d_2], [\log_\mu(1 - l_2), \log_\mu(1 - m_2), \log_\mu(1 - n_2), \log_\mu(1 - p_2)],$$
$$[\log_\mu(1 - x_2), \log_\mu(1 - y_2), \log_\mu(1 - v_2), \log_\mu(1 - w_2)]\rangle.$$

$$\therefore \log_\mu \widetilde{A}_1 \oplus \log_\mu \widetilde{A}_2$$
$$= \langle[1 - (\log_\mu a_1)(\log_\mu a_2), 1 - (\log_\mu b_1)(\log_\mu b_2), 1 - (\log_\mu c_1)(\log_\mu c_2), 1 - (\log_\mu d_1)(\log_\mu d_2)],$$
$$[\log_\mu(1 - l_1)\log_\mu(1 - l_2), \log_\mu(1 - m_1)\log_\mu(1 - m_2), \log_\mu(1 - n_1)\log_\mu(1 - n_2)\log_\mu(1 - p_1)\log_\mu(1 - p_2)],$$
$$[\log_\mu(1 - x_1)\log_\mu(1 - x_2), \log_\mu(1 - y_1)\log_\mu(1 - y_2), \log_\mu(1 - v_1)\log_\mu(1 - v_2), \log_\mu(1 - w_1)\log_\mu(1 - w_2)]\rangle.$$

Now for $k > 0$ we have,

$$k(\log_\mu \widetilde{A}_1 \oplus \log_\mu \widetilde{A}_2),$$
$$= \langle[1 - ((\log_\mu a_1)(\log_\mu a_2))^k, 1 - ((\log_\mu b_1)(\log_\mu b_2))^k, 1 - ((\log_\mu c_1)(\log_\mu c_2))^k, 1 - ((\log_\mu d_1)(\log_\mu d_2))^k],$$
$$[((\log_\mu(1 - l_1)\log_\mu(1 - l_2))^k, ((\log_\mu(1 - m_1)\log_\mu(1 - m_2))^k, ((\log_\mu(1 - n_1)\log_\mu(1 - n_2))^k,$$
$$((\log_\mu(1 - p_1)\log_\mu(1 - p_2))^k], [(((\log_\mu(1 - x_1)\log_\mu(1 - x_2))^k, ((\log_\mu(1 - y_1)\log_\mu(1 - y_2))^k,$$
$$((\log_\mu(1 - v_1)\log_\mu(1 - v_2))^k, ((\log_\mu(1 - w_1)\log_\mu(1 - w_2))^k]\rangle$$
$$= \langle[1 - (\log_\mu a_1)^k, 1 - (\log_\mu b_1)^k, 1 - (\log_\mu c_1)^k, 1 - (\log_\mu d_1)^k], [(\log_\mu(1 - l_1))^k, (\log_\mu(1 - m_1))^k,$$
$$(\log_\mu(1 - n_1))^k, (\log_\mu(1 - p_1))^k], [(\log_\mu(1 - x_1))^k, (\log_\mu(1 - y_1))^k, (\log_\mu(1 - v_1))^k, (\log_\mu(1 - w_1))^k]\rangle$$
$$\oplus \langle[1 - (\log_\mu a_2)^k, 1 - (\log_\mu b_2)^k, 1 - (\log_\mu c_2)^k, 1 - (\log_\mu d_2)^k], [(\log_\mu(1 - l_2))^k, (\log_\mu(1 - m_2))^k,$$
$$(\log_\mu(1 - n_2))^k, (\log_\mu(1 - p_2))^k], [(\log_\mu(1 - x_2))^k, (\log_\mu(1 - y_2))^k, (\log_\mu(1 - v_2))^k, (\log_\mu(1 - w_2))^k]\rangle$$
$$= k \log_\mu \widetilde{A}_1 \oplus k \log_\mu \widetilde{A}_2.$$

ii) This proof is similar to the previous one.

iii) For any $k_1, k_2 > 0$, we have

$$k_1 \log_\mu \widetilde{A}_1 \oplus k_2 \log_\mu \widetilde{A}_1$$
$$= \langle[1 - (\log_\mu a_1)^{k_1}, 1 - (\log_\mu b_1)^{k_1}, 1 - (\log_\mu c_1)^{k_1}, 1 - (\log_\mu d_1)^{k_1}], [(\log_\mu(1 - l_1))^{k_1}, (\log_\mu(1 - m_1))^{k_1},$$
$$(\log_\mu(1 - n_1))^{k_1}, (\log_\mu(1 - p_1))^{k_1}], [(\log_\mu(1 - x_1))^{k_1}, (\log_\mu(1 - y_1))^{k_1}, (\log_\mu(1 - v_1))^{k_1}, (\log_\mu(1 - w_1))^{k_1}]\rangle$$
$$\oplus \langle[1 - (\log_\mu a_2)^{k_1}, 1 - (\log_\mu b_2)^{k_1}, 1 - (\log_\mu c_2)^{k_2}, 1 - (\log_\mu d_2)^{k_2}], [(\log_\mu(1 - l_2))^{k_2}, (\log_\mu(1 - m_2))^{k_2},$$
$$(\log_\mu(1 - n_2))^{k_2}, (\log_\mu(1 - p_2))^{k_2}], [(\log_\mu(1 - x_2))^{k_2}, (\log_\mu(1 - y_2))^{k_2}, (\log_\mu(1 - v_2))^{k_2}, (\log_\mu(1 - w_2))^{k_2}]\rangle$$
$$= \langle[1 - (\log_\mu a_1)^{k_1+k_2}, 1 - (\log_\mu b_1)^{k_1+k_2}, 1 - (\log_\mu c_1)^{k_1+k_2}, 1 - (\log_\mu d_1)^{k_1+k_2}], [(\log_\mu(1 - l_1))^{k_1+k_2},$$
$$(\log_\mu(1 - m_1))^{k_1+k_2}, (\log_\mu(1 - n_1))^{k_1+k_2}, (\log_\mu(1 - p_1))^{k_1+k_2}], [(\log_\mu(1 - x_1))^{k_1+k_2}, (\log_\mu(1 - y_1))^{k_1+k_2},$$
$$(\log_\mu(1 - v_1))^{k_1+k_2}, (\log_\mu(1 - w_1))^{k_1+k_2}]\rangle$$
$$= (k_1 + k_2) \log_\mu \widetilde{A}_1.$$

iv) Again for any $k_1, k_2 > 0$, we get

$$
\begin{aligned}
&(\log_\mu \widetilde{A}_1)^{k_1} \bigotimes (\log_\mu \widetilde{A}_1)^{k_2} \\
&= \Big\langle \big[(1 - \log_\mu a_1)^{k_1}, (1 - \log_\mu b_1)^{k_1}, (1 - \log_\mu c_1)^{k_1}, (1 - \log_\mu d_1)^{k_1}\big], \big[1 - (1 - \log_\mu (1 - l_1))^{k_1}, \\
&\quad 1 - (1 - \log_\mu (1 - m_1))^{k_1}, 1 - (1 - \log_\mu (1 - n_1))^{k_1}, 1 - (1 - \log_\mu (1 - p_1))^{k_1}\big], \big[1 - (1 - \log_\mu (1 - x_1))^{k_1}, \\
&\quad 1 - (1 - \log_\mu (1 - y_1))^{k_1}, 1 - (1 - \log_\mu (1 - v_1))^{k_1}, 1 - (1 - \log_\mu (1 - w_1))^{k_1}\big] \Big\rangle \bigotimes \Big\langle \big[(1 - \log_\mu a_1)^{k_2}, (1 - \log_\mu b_1)^{k_2}, \\
&\quad (1 - \log_\mu c_1)^{k_2}, (1 - \log_\mu d_1)^{k_2}\big], \big[1 - (1 - \log_\mu (1 - l_1))^{k_2}, 1 - (1 - \log_\mu (1 - m_1))^{k_2}, 1 - (1 - \log_\mu (1 - n_1))^{k_2}, \\
&\quad 1 - (1 - \log_\mu (1 - p_1))^{k_2}\big], \big[1 - (1 - \log_\mu (1 - x_1))^{k_2}, 1 - (1 - \log_\mu (1 - y_1))^{k_2}, 1 - (1 - \log_\mu (1 - v_1))^{k_2}, \\
&\quad 1 - (1 - \log_\mu (1 - w_1))^{k_2}\big] \Big\rangle \\
&= \Big\langle \big[(1 - \log_\mu a_1)^{k_1 + k_2}, (1 - \log_\mu b_1)^{k_1 + k_2}, (1 - \log_\mu c_1)^{k_1 + k_2}, (1 - \log_\mu d_1)^{k_1 + k_2}\big], \\
&\quad \big[1 - (1 - \log_\mu (1 - l_1))^{k_1 + k_2}, 1 - (1 - \log_\mu (1 - m_1))^{k_1 + k_2}\big], \big[1 - (1 - \log_\mu (1 - n_1))^{k_1 + k_2}, \\
&\quad 1 - (1 - \log_\mu (1 - p_1))^{k_1 + k_2}\big], \big[1 - (1 - \log_\mu (1 - x_1))^{k_1 + k_2}, 1 - (1 - \log_\mu (1 - y_1))^{k_1 + k_2}, \\
&\quad 1 - (1 - \log_\mu (1 - v_1))^{k_1 + k_2}, 1 - (1 - \log_\mu (1 - w_1))^{k_1 + k_2}\big] \Big\rangle \\
&= (\log_\mu \widetilde{A}_1)^{k_1 + k_2}
\end{aligned}
$$

v) The proof of this part is trivial and hence omitted.

$\square$

4 Aggregation operators

In the decision making method, generally two types of aggregation operators are used namely weighted arithmetic operator and geometric averaging operator. Here, we proposed two new aggregation operators laws for TNN namely L_{arm} and L_{geo}, which are as follows:

Definition 4.1 Let $\widetilde{A}_t = \langle (a_t, b_t, c_t, d_t), (l_t, m_t, n_t, p_t), (x_t, y_t, v_t, w_t) \rangle$ $(t = 1, 2, \cdots, k)$ be any collection of TNNs and $0 < \mu \le \min (a_t, b_t, c_t, d_t, 1 - l_t, 1 - m_t, 1 - n_t, 1 - p_t, 1 - x_t, 1 - y_t, 1 - v_t, 1 - w_t) < 1$. The logarithmic trapezoidal neutrosophic weighted arithmetic aggregation operator $L_{arm} : \Gamma^k \to \Gamma$ is defined as

$$
L_{arm}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_k) = \phi_1 \log_{\mu_1} \widetilde{A}_1 \bigoplus \phi_2 \log_{\mu_2} \widetilde{A}_2 \bigoplus \cdots \bigoplus \phi_k \log_{\mu_k} \widetilde{A}_k,
$$

where $\omega = (\phi_1, \phi_2, \cdots, \phi_k)^T$ is the weight vector with $\phi_t \ge 0$ and $\sum_{t=1}^{k} \phi_t = 1$.

Note 4.1.1 For convenience, we denote $L_{arm}(A_1, \widetilde{A}_2, \cdots, \widetilde{A}_k) = L_{arm}$.

Theorem 4.1 Let $\widetilde{A}_s = \langle (a_s, b_s, c_s, d_s), (l_s, m_s, n_s, p_s), (x_s, y_s, v_s, w_s) \rangle$ $(s = 1, 2, \cdots, p)$ be any collection of TNNs. Then the aggregated value by using L_{arm} operator is also TNN and is given by

$$
L_{arm} = \begin{cases}
\Big\langle \big[1 - \prod_{s=1}^{p} (\log_{\mu_s} a_s)^{\phi_s}, 1 - \prod_{s=1}^{p} (\log_{\mu_s} b_s)^{\phi_s}, 1 - \prod_{s=1}^{p} (\log_{\mu_s} c_s)^{\phi_s}, 1 - \prod_{s=1}^{p} (\log_{\mu_s} d_s)^{\phi_s}\big], \\
\big[\prod_{s=1}^{p} (\log_{\mu_s} (1 - l_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\mu_s} (1 - m_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\mu_s} (1 - n_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\mu_s} (1 - p_s))^{\phi_s}\big], \\
\big[\prod_{s=1}^{p} (\log_{\mu_s} (1 - x_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\mu_s} (1 - y_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\mu_s} (1 - v_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\mu_s} (1 - w_s))^{\phi_s}\big] \Big\rangle; \\
0 < \mu_s \le \min (a_s, b_s, c_s, d_s, 1 - l_s, 1 - m_s, 1 - n_s, 1 - p_s, 1 - x_s, 1 - y_s, 1 - v_s, 1 - w_s) < 1 \\[6pt]
\Big\langle \big[1 - \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} a_s)^{\phi_s}, 1 - \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} b_s)^{\phi_s}, 1 - \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} c_s)^{\phi_s}, 1 - \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} d_s)^{\phi_s}\big], \\
\big[\prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} (1 - l_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} (1 - m_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} (1 - n_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} (1 - p_s))^{\omega_s}\big], \\
\big[\prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} (1 - x_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} (1 - y_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} (1 - v_s))^{\phi_s}, \prod_{s=1}^{p} (\log_{\frac{1}{\mu_s}} (1 - w_s))^{\omega_s}\big] \Big\rangle; \\
0 < \frac{1}{\mu_s} \le \min (a_s, b_s, c_s, d_s, 1 - l_s, 1 - m_s, 1 - n_s, 1 - p_s, 1 - x_s, 1 - y_s, 1 - v_s, 1 - w_s) < 1
\end{cases}
\tag{1}
$$

Proof To prove the Theorem 4.1, we use mathematical induction on s, where $0 < \mu_s \leq \min(a_s, b_s, c_s, d_s, 1 - l_s, 1 - m_s, 1 - n_s, 1 - p_s, 1 - x_s, 1 - y_s, 1 - v_s, 1 - w_s) < 1$ $(s = 1, 2, \cdots, p)$. When $s = 2$, we get

$$
\begin{aligned}
&L_{arm}(\widetilde{A}_1, \widetilde{A}_2)\\
&= \phi_1 \log_{\mu_1} \widetilde{A}_1 \oplus \phi_2 \log_{\mu_2} \widetilde{A}_2\\
&= \left\langle \left[1 - (\log_{\mu_1} a_1)^{\phi_1}, 1 - (\log_{\mu_1} b_1)^{\phi_1}, 1 - (\log_{\mu_1} c_1)^{\phi_1}, 1 - (\log_{\mu_1} d_1)^{\phi_1}\right], \left[(\log_{\mu_1}(1 - l_1))^{\phi_1}, (\log_{\mu_1}(1 - m_1))^{\phi_1}, \right.\right.\\
&\qquad \left.(\log_{\mu_1}(1 - n_1))^{\phi_1}, (\log_{\mu_1}(1 - p_1))^{\phi_1}\right], \left[(\log_{\mu_1}(1 - x_1))^{\phi_1}, (\log_{\mu_1}(1 - y_1))^{\phi_1}, (\log_{\mu_1}(1 - v_1))^{\phi_1}, (\log_{\mu_1}(1 - w_1))^{\phi_1}\right]\rangle\\
&\oplus \left\langle\left[1 - (\log_{\mu_2} a_2)^{\phi_2}, 1 - (\log_{\mu_2} b_2)^{\phi_2}, 1 - (\log_{\mu_2} c_2)^{\phi_2}, 1 - (\log_{\mu_2} d_2)^{\phi_2}\right], \left[(\log_{\mu_2}(1 - l_2))^{\phi_2}, (\log_{\mu_2}(1 - m_2))^{\phi_2},\right.\right.\\
&\qquad \left.(\log_{\mu_2}(1 - n_2))^{\phi_2}, (\log_{\mu_2}(1 - p_2))^{\phi_2}\right], \left[(\log_{\mu_2}(1 - x_2))^{\phi_2}, (\log_{\mu_2}(1 - y_2))^{\phi_2}, (\log_{\mu_2}(1 - v_2))^{\phi_2}, (\log_{\mu_2}(1 - w_2))^{\phi_2}\right]\rangle\\
&= \left\langle\left[1 - (\log_{\mu_1} a_1)^{\phi_1}(\log_{\mu_2} a_2)^{\phi_2}, 1 - (\log_{\mu_1} b_1)^{\phi_1}(\log_{\mu_2} b_2)^{\phi_2}, 1 - (\log_{\mu_1} c_1)^{\phi_1}(\log_{\mu_2} c_2)^{\phi_2}, 1 - (\log_{\mu_1} d_1)^{\phi_1}(\log_{\mu_2} d_2)^{\phi_2}\right],\right.\\
&\qquad \left[(\log_{\mu_1}(1 - l_1))^{\phi_1}(\log_{\mu_2}(1 - l_2))^{\phi_2}, (\log_{\mu_1}(1 - m_1))^{\phi_1}(\log_{\mu_2}(1 - m_2))^{\phi_2}, (\log_{\mu_1}(1 - n_1))^{\phi_1}(\log_{\mu_2}(1 - n_2))^{\phi_2},\right.\\
&\qquad \left.(\log_{\mu_1}(1 - p_1))^{\phi_1}(\log_{\mu_2}(1 - p_2))^{\phi_2}\right], \left[(\log_{\mu_1}(1 - x_1))^{\phi_1}(\log_{\mu_2}(1 - x_2))^{\phi_2}, (\log_{\mu_1}(1 - y_1))^{\phi_1}(\log_{\mu_2}(1 - y_2))^{\phi_2},\right.\\
&\qquad \left.\left.(\log_{\mu_1}(1 - v_1))^{\phi_1}(\log_{\mu_2}(1 - v_2))^{\phi_2}, (\log_{\mu_1}(1 - w_1))^{\phi_1}(\log_{\mu_2}(1 - w_2))^{\phi_2}\right]\right\rangle\\
&= \left\langle\left[1 - \prod_{s=1}^{2}(\log_{\mu_s} a_s)^{\phi_s}, 1 - \prod_{s=1}^{2}(\log_{\mu_s} b_s)^{\phi_s}, 1 - \prod_{s=1}^{2}(\log_{\mu_s} c_s)^{\phi_s}, 1 - \prod_{s=1}^{2}(\log_{\mu_s} d_s)^{\phi_s}\right],\right.\\
&\qquad \left[\prod_{s=1}^{2}(\log_{\mu_s}(1 - l_s))^{\phi_s}, \prod_{s=1}^{2}(\log_{\mu_s}(1 - m_s))^{\phi_s}, \prod_{s=1}^{2}(\log_{\mu_s}(1 - n_s))^{\phi_s}, \prod_{s=1}^{2}(\log_{\mu_s}(1 - p_s))^{\phi_s}\right],\\
&\qquad \left.\left[\prod_{s=1}^{2}(\log_{\mu_s}(1 - x_s))^{\phi_s}, \prod_{s=1}^{2}(\log_{\mu_s}(1 - y_s))^{\phi_s}, \prod_{s=1}^{2}(\log_{\mu_s}(1 - v_s))^{\phi_s}, \prod_{s=1}^{2}(\log_{\mu_s}(1 - w_s))^{\phi_s}\right]\right\rangle
\end{aligned}
$$

Thus, the Theorem is true for *s*=2. Let us assume that the Theorem is true for $s = p$. Then

$$
\begin{aligned}
&L_{arm}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p)\\
&= \left\langle[1 - \prod_{s=1}^{p}(\log_{\mu_s} a_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} b_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} c_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} d_s)^{\phi_s}\right], \left[\prod_{s=1}^{p}(\log_{\mu_s}(1 - l_s))^{\phi_s},\right.\\
&\qquad \prod_{s=1}^{p}(\log_{\mu_s}(1 - m_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - n_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - p_s))^{\phi_s}\right], \left[\prod_{s=1}^{p}(\log_{\mu_s}(1 - x_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - y_s))^{\phi_s},\right.\\
&\qquad \left.\left.\prod_{s=1}^{p}(\log_{\mu_s}(1 - v_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - w_s))^{\phi_s}\right]\right\rangle
\end{aligned}
$$

Now,

$$
\begin{aligned}
&L_{arm}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p, \widetilde{A}_{p+1})\\
&= \left\langle\left[1 - \prod_{s=1}^{p}(\log_{\mu_s} a_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} b_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} c_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} d_s)^{\phi_s}\right], \left[\prod_{s=1}^{p}(\log_{\mu_s}(1 - l_s))^{\phi_s},\right.\right.\\
&\qquad \prod_{s=1}^{p}(\log_{\mu_s}(1 - m_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - n_s))^{\phi_s}, \prod_{s=1}^{m}(\log_{\mu_s}(1 - p_s))^{\phi_s}\right], \left[\prod_{s=1}^{p}(\log_{\mu_s}(1 - x_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - y_s))^{\phi_s},\right.\\
&\qquad \left.\left.\prod_{s=1}^{p}(\log_{\mu_s}(1 - v_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - w_s))^{\phi_s}\right]\right\rangle \oplus \phi_{p+1} \log_{\mu_{p+1}} \widetilde{A}_{p+1}\\
&= \left\langle\left[1 - \prod_{s=1}^{p}(\log_{\mu_s} a_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} b_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} c_s)^{\phi_s}, 1 - \prod_{s=1}^{p}(\log_{\mu_s} d_s)^{\phi_s}\right], \left[\prod_{s=1}^{p}(\log_{\mu_s}(1 - l_s))^{\phi_s},\right.\right.\\
&\qquad \prod_{s=1}^{p}(\log_{\mu_s}(1 - m_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - n_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - p_s))^{\phi_s}\right], \left[\prod_{s=1}^{p}(\log_{\mu_s}(1 - x_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1 - y_s))^{\phi_s},\right.
\end{aligned}
$$

$$\prod_{s=1}^{p}(\log_{\mu_s}(1-v_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1-w_s))^{\phi_s}\Big]\Big\rangle \oplus \Big\langle\Big[1-\overline{(\log_{\mu_{p+1}}a_{p+1})^{\phi_{p+1}}}, 1-(\log_{\mu_{p+1}}b_{p+1})^{\phi_{p+1}},$$

$$\prod_{s=1}^{p}1-(\log_{\mu_{p+1}}c_{p+1})^{\phi_{p+1}}, 1-(\log_{\mu_{p+1}}d_{p+1})^{\phi_{p+1}}\Big], \Big[(\log_{\mu_{p+1}}(1-l_{p+1}))^{\phi_{p+1}}, (\log_{\mu_{p+1}}(1-m_{p+1}))^{\phi_{p+1}},$$

$$(\log_{\mu_{p+1}}(1-n_{p+1}))^{\phi_{p+1}}, (\log_{\mu_{p+1}}(1-p_{p+1}))^{\phi_{p+1}}\Big], \Big[(\log_{\mu_{p+1}}(1-x_{p+1}))^{\phi_{p+1}}, (\log_{\mu_{p+1}}(1-y_{p+1}))^{\phi_{p+1}},$$

$$(\log_{\mu_{p+1}}(1-v_{p+1}))^{\phi_{p+1}}, (\log_{\mu_{p+1}}(1-w_{p+1}))^{\phi_1}\Big]\Big\rangle$$

$$= \Big\langle\Big[1-\prod_{s=1}^{p+1}(\log_{\mu_s}a_s)^{\phi_s}, 1-\prod_{s=1}^{p+1}(\log_{\mu_s}b_s)^{\phi_s}, 1-\prod_{s=1}^{p}(\log_{\mu_s}c_s)^{\phi_s}, 1-\prod_{s=1}^{p+1}(\log_{\mu_s}d_s)^{\phi_s}\Big], \Big[\prod_{s=1}^{p+1}(\log_{\mu_s}(1-l_s))^{\phi_s},$$

$$\prod_{s=1}^{p+1}(\log_{\mu_s}(1-m_s))^{\phi_s}, \prod_{s=1}^{p+1}(\log_{\mu_s}(1-n_s))^{\phi_s}, \prod_{s=1}^{p+1}(\log_{\mu_s}(1-p_s))^{\phi_s}\Big], \Big[\prod_{s=1}^{p+1}(\log_{\mu_s}(1-x_s))^{\phi_s}, \prod_{s=1}^{p+1}(\log_{\mu_s}(1-y_s))^{\phi_s},$$

$$\prod_{s=1}^{p+1}(\log_{\mu_s}(1-v_s))^{\phi_s}, \prod_{s=1}^{p+1}(\log_{\mu_s}(1-w_s))^{\phi_s}]\Big\rangle$$

This shows that the Theorem is valid for s=p+1. Hence by mathematical induction, we can say that the above Theorem holds for all integral value of s.

Again, if $0 < \frac{1}{\mu_s} \le \min(a_s, b_s, c_s, d_s, 1-l_s, 1-m_s, 1-n_s, 1-p_s, 1-x_s, 1-y_s, 1-v_s, 1-w_s) < 1$, then proceeding in the similar approach as in above case, we also get

$$L_{arm}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p)$$
$$= \Big\langle\Big[1-\prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}a_s)^{\phi_s}, 1-\prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}b_s)^{\phi_s}, 1-\prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}c_s)^{\phi_s}, 1-\prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}d_s)^{\phi_s}\Big],$$
$$\Big[\prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}(1-l_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}(1-m_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}(1-n_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}(1-p_s))^{\omega_s}\Big],$$
$$\Big[\prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}(1-x_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}(1-y_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}(1-v_s))^{\phi_s}, \prod_{s=1}^{p}(\log_{\frac{1}{\mu_s}}(1-w_s))^{\omega_s}\Big]\Big\rangle.$$

$\square$

4.1 Properties of aggregation operator

In this subsection the Properties of L_{arm} operator has been presented. Here, it is assumed that $\mu_1 = \mu_2 = \cdots = \mu_p = \mu$ (say) and $0 < \mu \le \min(a_s, b_s, c_s, d_s, 1-l_s, 1-m_s, 1-n_s, 1-p_s, 1-x_s, 1-y_s, 1-v_s, 1-w_s) < 1$. Also $\omega = (\phi_1, \phi_2, \cdots, \phi_p)^T$ be the weight vector such that $\phi_s \ge 0$ and $\sum_{s=1}^{p}\phi_s = 1$.

Lemma 4.1.1 (Idempotency of L_{arm} operator) *If* $\widetilde{A}_s = \widetilde{A}$, $\forall s$, *where* $\widetilde{A} = \langle(a, b, c, d), (l, m, n, p), (x, y, v, w)\rangle$ *then*
$$L_{arm}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p) = \log_\mu \widetilde{A}$$

Proof Since $\widetilde{A}_s = \widetilde{A}$, $\forall$ s, where $\widetilde{A} = \langle(a, b, c, d), (l, m, n, p), (x, y, v, w)\rangle$ is an TNN such that $\widetilde{A}_s = \widetilde{A}$, $\forall s$. Then, from the Theorem (4.1), we get

$$L_{arm}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p)$$
$$= \Big\langle\Big[1-\prod_{s=1}^{p}(\log_{\mu_s}a)^{\phi_s}, 1-\prod_{s=1}^{p}(\log_{\mu_s}b)^{\phi_s}, 1-\prod_{s=1}^{p}(\log_{\mu_s}c)^{\phi_s}, 1-\prod_{s=1}^{p}(\log_{\mu_s}d)^{\phi_s}\Big], \Big[\prod_{s=1}^{p}(\log_{\mu_s}(1-l))^{\phi_s},$$
$$\prod_{s=1}^{p}(\log_{\mu_s}(1-m))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1-n))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1-p))^{\phi_s}\Big], \Big[\prod_{s=1}^{p}(\log_{\mu_s}(1-x))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1-y))^{\phi_s},$$
$$\prod_{s=1}^{p}(\log_{\mu_s}(1-v))^{\phi_s}, \prod_{s=1}^{p}(\log_{\mu_s}(1-w))^{\phi_s}\Big]\Big\rangle$$

$$= \left\langle \left[1 - (\log_\mu a)^{\sum \phi_s}, 1 - (\log_\mu b)^{\sum \phi_s}, 1 - (\log_\mu c)^{\sum \phi_s}, 1 - \frac{(\log_\mu d)^{\sum \phi_s}}{(\log_\mu (1-y))^{\sum \phi_s}}, \frac{(\log_\mu (1-l))^{\sum \phi_s}, (\log_\mu (1-m))^{\sum \phi_s}}{(\log_\mu (1-v))^{\sum \phi_s}}, \right. \right.$$
$$(\log_\mu (1-n))^{\sum \phi_s}, (\log_\mu (1-p))^{\sum \phi_s}], [(\log_\mu (1-x))^{\sum \phi_s}, \frac{}{(\log_\mu (1-y))^{\sum \phi_s}, (\log_\mu (1-v))^{\sum \phi_s}},$$
$$\left. \left. (\log_\mu (1-w))^{\sum \phi_s} \right] \right\rangle, \quad \left[\text{ since, } \delta_s = \delta \text{ and } \widetilde{A}_s = \widetilde{A} \ \forall \ s \right]$$
$$= \log_\mu \widetilde{A}.$$

□

Lemma 4.1.2 (Boundedness of L_{arm} operator) *Let* $\widetilde{A}_s = \langle (a_s, b_s, c_s, d_s), (l_s, m_s, n_s, p_s), (x_s, y_s, v_s, w_s) \rangle$, $(s = 1, 2, \cdots, p)$ *be any collection of TNNs and let*
$$\widetilde{A}_{min} = \langle [\min a_s, \min b_s, \min c_s, \min d_s], [\max l_s, \max m_s,$$
$$\max n_s, \max p_s], [\max x_s, \max y_s, \max v_s, \max w_s] \rangle,$$
$$\widetilde{A}_{max} = \langle [\max a_s, \max b_s, \max c_s, \max d_s], [\min l_s, \min$$
$$m_s, \min n_s, \min p_s], [\min x_s, \min y_s, \min v_s, \min w_s] \rangle,$$
$$\widetilde{A}^- = L_{arm}(\widetilde{A}_{min}, \widetilde{A}_{min}, \cdots, \widetilde{A}_{min}),$$
$$\widetilde{A}^+ = L_{arm}(\widetilde{A}_{max}, \widetilde{A}_{max}, \cdots, \widetilde{A}_{max}).$$

Then we have
$$\widetilde{A}^- \le L_{arm}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p) \le \widetilde{A}^+$$

Proof The proof of the Lemma follows from the Theorem 4.1 and the Lemma 4.1.1. □

Lemma 4.1.3 (Monotonicity of L_{arm} operator) *Let* $\widetilde{A}_s = \langle (a_s, b_s, c_s, d_s), (l_s, m_s, n_s, p_s), (x_s, y_s, v_s, w_s) \rangle$ *and* $\widetilde{A}'_s = \langle (a'_s, b'_s, c'_s, d'_s), (l'_s, m'_s, n'_s, p'_s), (x'_s, y'_s, v'_s, w'_s) \rangle$, $(s = 1, 2, \cdots, p)$ *be two collection of TNNs. If* $\widetilde{A}_s \le \widetilde{A}'_s \ \forall \ s$, *then*
$$L_{arm}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p) \le L_{arm}(\widetilde{A}'_1, \widetilde{A}'_2, \cdots, \widetilde{A}'_p)$$

Proof The proof of above Lemma is similar to the Lemma 4.1.2 and hence omitted. □

Definition 4.2 Let $\widetilde{A}_s = \langle (a_s, b_s, c_s, d_s), (l_s, m_s, n_s, p_s), (x_s, y_s, v_s, w_s) \rangle$, $(s = 1, 2, \cdots, p)$ be any collection of TNNs and $0 < \mu \le \min(a_s, b_s, c_s, d_s, 1-l_s, 1-m_s, 1-n_s, 1-p_s, 1-x_s, 1-y_s, 1-v_s, 1-w_s) < 1$. The logarithmic trapezoidal neutrosophic weighted geometric aggregation operator $L_{geo} : \Gamma^p \to \Gamma$ is defined as

$$L_{geo}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p) = \left(\log_{\mu_1} \widetilde{A}_1 \right)^{\phi_1} \bigotimes \left(\log_{\mu_2} \widetilde{A}_2 \right)^{\phi_2}$$
$$\bigotimes \cdots \bigotimes \left(\log_{\mu_p} \widetilde{A}_p \right)^{\phi_p}$$

where $\omega = (\phi_1, \phi_2, \cdots, \phi_p)^T$ is the weight vector with $\phi_s \ge 0$ and $\sum_{s=1}^{p} \phi_s = 1$.

Theorem 4.2 Let $\widetilde{A}_s = \langle (a_s, b_s, c_s, d_s), (l_s, m_s, n_s, p_s), (x_s, y_s, v_s, w_s) \rangle$ $(s = 1, 2, \cdots, p)$ be any collection of TNNs. Then the aggregated value by using L_{geo} operator is also TNN and is given by

$$L_{geo} = \begin{cases} \left\langle \left[\prod_{s=1}^{p} (1 - \log_{\mu_s} a_s)^{\phi_s}, \prod_{s=1}^{p} (1 - \log_{\mu_s} b_s)^{\phi_s}, \prod_{s=1}^{p} (1 - \log_{\mu_s} c_s)^{\phi_s}, \prod_{s=1}^{p} (1 - \log_{\mu_s} d_s)^{\phi_s} \right], \\ \left[1 - \prod_{s=1}^{p} (1 - \log_{\mu_s} (1-l_s))^{\phi_s}, 1 - \prod_{s=1}^{p} (1 - \log_{\mu_s} (1-m_s))^{\phi_s}, 1 - \prod_{s=1}^{p} (1 - \log_{\mu_s} (1-n_s))^{\phi_s}, \\ 1 - \prod_{s=1}^{p} (1 - \log_{\mu_s} (1-p_s))^{\phi_s} \right], \left[1 - \prod_{s=1}^{p} (1 - \log_{\mu_s} (1-x_s))^{\phi_s}, 1 - \prod_{s=1}^{p} (1 - \log_{\mu_s} (1-y_s))^{\phi_s}, \\ 1 - \prod_{s=1}^{p} (1 - \log_{\mu_s} (1-v_s))^{\phi_s}, 1 - \prod_{s=1}^{p} (1 - \log_{\mu_s} (1-w_s))^{\phi_s} \right] \right\rangle; \\ 0 < \mu_s \le \min(a_s, b_s, c_s, d_s, 1-l_s, 1-m_s, 1-n_s, 1-p_s, 1-x_s, 1-y_s, 1-v_s, 1-w_s) < 1 \\ \left\langle \left[\prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} a_s)^{\phi_s}, \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} b_s)^{\phi_s}, \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} c_s)^{\phi_s}, \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} d_s)^{\phi_s} \right], \\ \left[1 - \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} (1-l_s))^{\phi_s}, 1 - \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} (1-m_s))^{\phi_s}, 1 - \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} (1-n_s))^{\phi_s}, \\ 1 - \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} (1-p_s))^{\phi_s} \right], \left[1 - \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} (1-x_s))^{\phi_s}, 1 - \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} (1-y_s))^{\phi_s}, \\ 1 - \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} (1-v_s))^{\phi_s}, 1 - \prod_{s=1}^{p} (1 - \log_{\frac{1}{\mu_s}} (1-w_s))^{\phi_s} \right] \right\rangle; \\ 0 < \frac{1}{\mu_s} \le \min(a_s, b_s, c_s, d_s, 1-l_s, 1-m_s, 1-n_s, 1-p_s, 1-x_s, 1-y_s, 1-v_s, 1-w_s) < 1 \end{cases} \quad (2)$$

Proof The proof of the Theorem is exactly same as Theorem 4.1 and hence omitted. □

Note 4.2.1 For convenience, we denote $L_{geo}(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_p) = L_{geo}$.

5 MCGDM technique based on L_{arm} and L_{geo} operators

MCGDM is a branch of operational research. In MCGDM technique, a group of expert/decision-makers are involve to select the best alternative from a given set of feasible alternatives with respect to some given criteria. Here, we have introduced an MCGDM technique by utilizing the operators L_{arm} & L_{geo}, scalar multiplication & addition of TNNs and its defuzzyfication method. In this technique, we have considered the influence of the decision makers weights in the decision making procedure. Here, we have considered the MCGDM technique as follows:

Let $U = \{U_1, U_2, \cdots, U_u\}$ be the set of 'u' different alternatives and $V = \{V_1, V_2, \cdots, V_v\}$ be the set of 'v' different attributes with the associated weight vectors $\omega = (\phi_1, \phi_2, \cdots, \phi_v)^T$, where $\phi_t \geq 0$ and $\sum_{t=1}^{v} \phi_t = 1$. Also, we take the set of decision-makers $W = \{W_1, W_2, \cdots, W_w\}$ whose weight values are assumed as $\Omega = \{\Omega_1, \Omega_2, \cdots, \Omega_w\}$, where $\Omega_k \geq 0, (k = 1, 2, \cdots, w)$ and satisfy the condition $\sum_{k=1}^{w} \Omega_k = 1$. Here, the weight values of the decision-maker's will be assumed according to ability of judgement, thinking ability, knowledge power, etc. According to the suitable judgement of the decision-makers, firstly we have constructed the decision matrices related with different alternatives. The evaluated values for the alternatives on the attributes are given as

$$\widetilde{A}^r_{ij} = \Big\langle \Big(a^r_{ij}, b^r_{ij}, c^r_{ij}, d^r_{ij}\Big), \Big(l^r_{ij}, m^r_{ij}, n^r_{ij}, p^r_{ij}\Big), \Big(x^r_{ij}, y^r_{ij}, v^r_{ij}, w^r_{ij}\Big)\Big\rangle, \; i = 1, 2, \cdots, u, \; j = 1, 2, \cdots, v, \; r = 1, 2, \cdots, w.$$

The associated decision matrix (DM) is characterized as follows:

$$\mathbf{DM^r} = \begin{array}{c} \\ U_1 \\ U_2 \\ \vdots \\ U_u \end{array} \begin{array}{c} V_1 \quad\;\; V_2 \;\; \cdots \;\; V_v \\ \begin{pmatrix} \widetilde{A}^r_{11} & \widetilde{A}^r_{12} & \cdots & \widetilde{A}^r_{1v} \\ \widetilde{A}^r_{21} & \widetilde{A}^r_{22} & \cdots & \widetilde{A}^r_{2v} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{A}^r_{u1} & \widetilde{A}^r_{u2} & \cdots & \widetilde{A}^r_{uv} \end{pmatrix} \end{array},$$

where $r = 1, 2, \cdots, w$.

Let the logarithmic base index for TNNs are given by μ^r_{ij} $(i = 1, 2, \cdots, u), (j = 1, 2, \cdots, v)$ where $0 < \mu^r_{ij} \leq$

min $(a_{ij}, b_{ij}, c_{ij}, d_{ij}, 1 - l_{ij}, 1 - m_{ij}, 1 - n_{ij}, 1 - p_{ij}, 1 - x_{ij}, 1 - y_{ij}, 1 - v_{ij}, 1 - w_{ij}) < 1$ which are summarised in the matrix form as follows:

$$\Omega^{\mathbf{r}} = \begin{array}{c} \\ U_1 \\ U_2 \\ \vdots \\ U_u \end{array} \begin{array}{c} V_1 \quad\;\; V_2 \;\; \cdots \;\; V_v \\ \begin{pmatrix} \mu^r_{11} & \mu^r_{12} & \cdots & \mu^r_{1v} \\ \mu^r_{21} & \mu^r_{22} & \cdots & \mu^r_{2v} \\ \vdots & \vdots & \vdots & \vdots \\ \mu^r_{u1} & \mu^r_{u2} & \cdots & \mu^r_{uv} \end{pmatrix} \end{array},$$

where $r = 1, 2, \cdots, w$.

Now, our MCGDM technique under TN environment has been executed through the following steps:

Step 1: Firstly, we apply the L_{arm} or L_{geo} operator on every decision matrix DM^r to get a new column matrix $C^r_{u \times 1}$ as follows

$$C^r_{u \times 1} = TNWEA(\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_v)$$
$$= \begin{array}{c} U_1 \\ U_2 \\ \vdots \\ U_u \end{array} \begin{pmatrix} \widetilde{A}^r_{11} \\ \widetilde{A}^r_{21} \\ \vdots \\ \widetilde{A}^r_{u1} \end{pmatrix},$$

where entities of column matrix $C^r_{u \times 1}$ is the aggregated evaluation values with respect to different criterion $(r = 1, 2, \cdots, w)$.

Step 2: Here, we obtain overall attribute values $\widetilde{A}_{s1}$ corresponding to the alternatives U_s $(s = 1, 2, \cdots, u)$ after utilizing decision-maker's (Ω_k) weights according to the relation $\sum_{k=1}^{w} \Omega_k C^k_{u \times 1}$ (scalar multiplication and addition of TNNs) in the form of final decision matrix (DM) as follows

$$\mathbf{DM} = \begin{array}{c} U_1 \\ U_2 \\ \vdots \\ U_u \end{array} \begin{pmatrix} \widetilde{A}_{11} \\ \widetilde{A}_{21} \\ \vdots \\ \widetilde{A}_{u1} \end{pmatrix}.$$

Step 3: We calculate $D_{Neu}(\widetilde{A}_{s1})$ of the alternatives U_s, $(s = 1, 2, \cdots, u)$ utilizing de-Neutrosophication technique according to the Definition 2.1.1.

Step 4: After getting all the de-Neutrosophication values of the corresponding alternatives, the alternatives have been ranked according to the Definition 2.1.2 and select the best one.

Remark 5.1 The steps of MCGDM technique have been shown pictorially in Fig. 2.

Fig. 2 Flowchart of our MCGDM technique
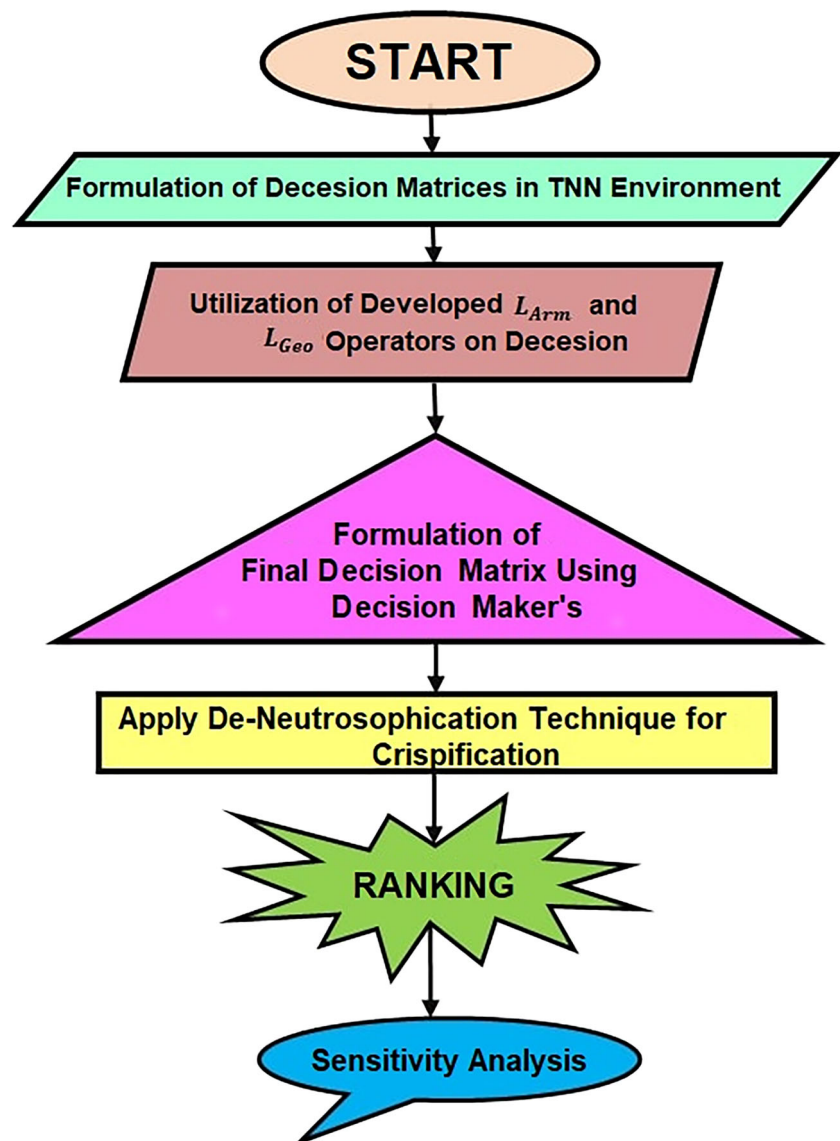
6 Detection of most harmful virus by utilizing proposed MCGDM technique

Let us consider a realistic problem linked with medical domain due to presence of disjunctive kinds of virus in our environment. In this current era, we observed that, humans of our world are suffering from many diseases and they deal with disjunctive sort of symptoms in exclusive times. It is a burning issue to identify which virus is the most harmful virus for human in recent times. Peoples always went to the hospital or nursing home and meet the doctor's for advice. Now the doctor's always try to identify the fever according to lab test report and symptoms on the patient's body. But, sometimes their minds are in dilemma about the virus and symptoms when they are so closely related to each other. Thus it is a problem of uncertainty domain in which neutrosophic components are present. People of our society are come to know about the virus and its effects according to the opinions of doctors. Now, our problem is to collect

data's from different doctors (Junior, Adult, Senior) related with virus and symptoms and create decision matrices in hesitation arena and focus to find out the most harmful virus in our environment. Thus, it becomes an MCGDM problem having three alternatives, three attributes and three types of decision-maker.

Let the alternatives are: U_1 = Virus 1 (Ebola Virus), U_2 = Virus 2 (Marburg Virus), U_3 = Virus 3 (Corona Virus) and the corresponding attributes are V_1 = Symptom 1 (Vomiting), V_2 = Symptom 2 (Sore Throat Problem), V_3 = Symptom 3 (cough and Red Eyes). Let us consider the decision-makers W_1 = Junior Doctor, W_2 = Adult Doctor, W_3 = Senior Doctor having weight value $D = \{0.33, 0.37, 0.3\}$ and the weight corresponding to the attribute function is taken as $\Delta = \{0.32, 0.35, 0.33\}$. The three alternatives are to be evaluated under these three attributes and give their preferences in terms of TNNs by the decision-makers.

The evaluated information of the alternatives U_i, $(i = 1, 2, \cdots, u)$ under the attribute V_j, $(j = 1, 2, \cdots, v)$ are characterized in the following trapezoidal neutrosophic number decision matrices:

$$\mathbf{DM^1} = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \begin{pmatrix} V_1 \\ \langle(0.4, 0.8, 0.8, 0.8), (0.6, 0.7, 0.8, 0.8), (0.1, 0.1, 0.2, 0.3)\rangle \\ \langle(0.4, 0.5, 0.5, 0.6), (0.7, 0.7, 0.9, 0.9), (0.1, 0.3, 0.5, 0.4)\rangle \\ \langle(0.7, 0.8, 0.9, 0.9), (0.4, 0.6, 0.8, 0.8), (0.3, 0.4, 0.6, 0.7)\rangle \end{pmatrix}$$

$$\begin{array}{c} V_2 \\ \langle(0.3, 0.4, 0.5, 0.5), (0.3, 0.6, 0.6, 0.7), (0.4, 0.6, 0.6, 0.7)\rangle \\ \langle(0.4, 0.6, 0.6, 0.7), (0.4, 0.4, 0.5, 0.5), (0.1, 0.3, 0.5, 0.5)\rangle \\ \langle(0.7, 0.8, 0.8, 0.9), (0.2, 0.3, 0.3, 0.3), (0.3, 0.3, 0.3, 0.4)\rangle \end{array}$$

$$\begin{array}{c} V_3 \\ \langle(0.4, 0.5, 0.6, 0.8), (0.2, 0.2, 0.2, 0.4), (0.3, 0.5, 0.6, 0.7)\rangle \\ \langle(0.5, 0.5, 0.6, 0.8), (0.5, 0.5, 0.6, 0.6), (0.1, 0.5, 0.6, 0.6)\rangle \\ \langle(0.7, 0.7, 0.8, 0.8), (0.2, 0.3, 0.4, 0.4), (0.1, 0.2, 0.3, 0.3)\rangle \end{array}$$

$$\mathbf{DM^2} = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \begin{pmatrix} V_1 \\ \langle(0.6, 0.7, 0.7, 0.8), (0.3, 0.4, 0.4, 0.5), (0.1, 0.4, 0.4, 0.4)\rangle \\ \langle(0.6, 0.6, 0.7, 0.7), (0.3, 0.4, 0.6, 0.6), (0.5, 0.6, 0.6, 0.7)\rangle \\ \langle(0.5, 0.7, 0.8, 0.8), (0.4, 0.4, 0.5, 0.5), (0.5, 0.5, 0.6, 0.6)\rangle \end{pmatrix}$$

$$\begin{array}{c} V_2 \\ \langle(0.5, 0.7, 0.9, 0.9), (0.3, 0.4, 0.4, 0.6), (0.5, 0.5, 0.6, 0.6)\rangle \\ \langle(0.4, 0.4, 0.5, 0.6), (0.5, 0.6, 0.6, 0.7), (0.2, 0.4, 0.5, 0.7)\rangle \\ \langle(0.4, 0.7, 0.7, 0.8), (0.4, 0.5, 0.6, 0.6), (0.5, 0.5, 0.5, 0.5)\rangle \end{array}$$

$$\begin{array}{c} V_3 \\ \langle(0.6, 0.8, 0.9, 0.9), (0.2, 0.2, 0.3, 0.4), (0.1, 0.3, 0.3, 0.4)\rangle \\ \langle(0.6, 0.7, 0.9, 0.9), (0.3, 0.3, 0.4, 0.4), (0.3, 0.3, 0.3, 0.3)\rangle \\ \langle(0.6, 0.9, 0.9, 0.9), (0.2, 0.2, 0.2, 0.3), (0.4, 0.4, 0.5, 0.5)\rangle \end{array}$$

$$\mathbf{DM^3} = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \begin{pmatrix} V_1 \\ \langle(0.4, 0.6, 0.8, 0.9), (0.2, 0.4, 0.6, 0.7), (0.3, 0.5, 0.7, 0.9)\rangle \\ \langle(0.8, 0.8, 0.9, 0.9), (0.1, 0.3, 0.5, 0.5), (0.5, 0.6, 0.6, 0.6)\rangle \\ \langle(0.5, 0.6, 0.7, 0.7), (0.3, 0.3, 0.4, 0.5), (0.2, 0.3, 0.3, 0.4)\rangle \end{pmatrix}$$

$$\begin{array}{c} V_2 \\ \langle(0.7, 0.7, 0.8, 0.8), (0.2, 0.2, 0.4, 0.4), (0.6, 0.6, 0.6, 0.7)\rangle \\ \langle(0.6, 0.6, 0.7, 0.8), (0.3, 0.3, 0.3, 0.3), (0.4, 0.4, 0.5, 0.5)\rangle \\ \langle(0.7, 0.7, 0.8, 0.9), (0.1, 0.2, 0.3, 0.3), (0.3, 0.3, 0.3, 0.4)\rangle \end{array}$$

$$\begin{array}{c} V_3 \\ \langle(0.7, 0.9, 0.9, 0.9), (0.3, 0.5, 0.6, 0.6), (0.4, 0.4, 0.5, 0.5)\rangle \\ \langle(0.8, 0.8, 0.8, 0.9), (0.2, 0.3, 0.3, 0.4), (0.2, 0.2, 0.4, 0.4)\rangle \\ \langle(0.7, 0.7, 0.7, 0.9), (0.2, 0.2, 0.3, 0.3), (0.2, 0.2, 0.2, 0.3)\rangle \end{array}$$

Furthermore, the logarithmic base matrices of corresponding decision matrices are characterized as

$$\Omega^1 = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \begin{array}{c} V_1 \quad V_2 \quad V_3 \\ \begin{pmatrix} 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.4 \\ 0.2 & 0.6 & 0.6 \end{pmatrix} \end{array},$$

$$\Omega^2 = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \begin{array}{c} V_1 \quad V_2 \quad V_3 \\ \begin{pmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \\ 0.4 & 0.3 & 0.5 \end{pmatrix} \end{array},$$

$$\Omega^3 = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \end{array} \begin{array}{c} V_1 \quad V_2 \quad V_3 \\ \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.5 & 0.6 \\ 0.5 & 0.6 & 0.7 \end{pmatrix} \end{array}.$$

Now, we have used the proposed technique under TN environment as follows:

Step 1: Firstly, we use the L_{arm} operator on each decision matrix DM^r according to the equation (1) we have new column matrices $C^r_{3\times1}$ ($r = 1, 2, 3$) as follows

$$\mathbf{C^1_{3\times1}} = \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} \begin{pmatrix} \langle(0.3918, 0.7694, 0.7743, 0.7893), (0.2910, 0.3385, 0.3476, 0.3871), (0.244, 0.3208, 0.1988, 0.3331)\rangle \\ \langle(0.3808, 0.5407, 0.5420, 0.6125), (0.5126, 0.6103, 0.6805, 0.7422), (0.0944, 0.3979, 0.4810, 0.5483)\rangle \\ \langle(0.6726, 0.7638, 0.7826, 0.7905), (0.3944, 0.5541, 0.5755, 0.6033), (0.3667, 0.4647, 0.6541, 0.7051)\rangle \end{pmatrix},$$

$$\mathbf{C^2_{3\times1}} = \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} \begin{pmatrix} \langle(0.1775, 0.5579, 0.7214, 0.7517), (0.4421, 0.4886, 0.5037, 1.000), (0.3250, 0.7206, 0.7281, 0.7392)\rangle \\ \langle(0.4970, 0.5352, 0.696, 0.6406), (0.4562, 0.5515, 0.5803, 0.6293), (0.4089, 0.4185, 0.4271, 0.7255)\rangle \\ \langle(0.2484, 0.7406, 0.7768, 0.7573), (0.4227, 0.4427, 0.4623, 0.5397), (0.4269, 0.4952, 0.6144, 0.7539)\rangle \end{pmatrix},$$

$$\mathbf{C^3_{3\times1}} = \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} \begin{pmatrix} \langle(0.5616, 0.7568, 0.8417, 0.8529), (0.2368, 0.3843, 0.6788, 0.7408), (0.5078, 0.6911, 0.7518, 0.8314)\rangle \\ \langle(0.6192, 0.6198, 0.6795, 0.7452), (0.3422, 0.6367, 0.6438, 0.7248), (0.6838, 0.6838, 0.8295, 0.8438)\rangle \\ \langle(0.2446, 0.3002, 0.395, 0.6889), (0.3986, 0.4046, 0.7998, 0.8819), (0.5960, 0.6107, 0.6146, 0.6787)\rangle \end{pmatrix}.$$

Again, if we utilize the operator L_{geo} operator according to the (2) on every decision matrix DM^r, we get new column matrices $C^r_{3\times1}$ ($r = 1, 2, 3$) as follows:

Table 1 Sensitivity analysis under L_{arm} operator

Decision makers weight	Final decision matrix	Ranking order
< 0.33, 0.37, 0.3 >	⟨(0.3836, 0.7019, 0.7575, 0.7615), (0.3193, 0.4179, 0.5298, 0.6054), (0.3380, 0.4476, 0.5765, 0.5886)⟩ ⟨(0.5044, 0.5680, 0.5910, 0.6677), (0.5063, 0.6009, 0.6350, 0.6995), (0.3069, 0.3727, 0.5138, 0.5494)⟩ ⟨(0.4278, 0.6474, 0.6789, 0.6819), (0.4059, 0.4902, 0.5858, 0.6171), (0.4731, 0.5164, 0.5533, 0.5951)⟩	$U_3 > U_1 > U_2$
< 0.36, 0.3, 0.34 >	⟨(0.4043, 0.7146, 0.7623, 0.7624), (0.3076, 0.4108, 0.5343, 0.5762), (0.3412, 0.4331, 0.4594, 0.5774)⟩ ⟨(0.5069, 0.5108, 0.577, 0.6715), (0.5093, 0.6070, 0.6306, 0.6598), (0.2974, 0.4821, 0.5224, 0.5349)⟩ ⟨(0.4418, 0.5322, 0.5481, 0.6730), (0.4041, 0.4941, 0.6027, 0.6183), (0.4920, 0.5198, 0.5452, 0.5817)⟩	$U_3 > U_1 > U_2$
< 0.33, 0.32, 0.35 >	⟨(0.4026, 0.7107, 0.7643, 0.7616), (0.3095, 0.4129, 0.5378, 0.5964), (0.3456, 0.4461, 0.4776, 0.5921)⟩ ⟨(0.5113, 0.5764, 0.5799, 0.6733), (0.4991, 0.6062, 0.6333, 0.6528), (0.3131, 0.4850, 0.5433, 0.5461)⟩ ⟨(0.4277, 0.5270, 0.5573, 0.6780), (0.4047, 0.4880, 0.5021, 0.5299), (0.5638, 0.5818, 0.5819, 0.5920)⟩	$U_3 > U_1 > U_2$
< 0.28, 0.35, 0.37 >	⟨(0.4011, 0.7046, 0.7680, 0.7602), (0.3121, 0.4162, 0.5442, 0.6310), (0.3538, 0.5684, 0.5998, 0.6176)⟩ ⟨(0.5190, 0.5695, 0.5840, 0.6768), (0.4821, 0.6052, 0.6376, 0.6586), (0.3410, 0.4907, 0.5306, 0.5652)⟩ ⟨(0.4033, 0.6167, 0.6708, 0.6858), (0.4057, 0.4777, 0.5021, 0.5508), (0.5034, 0.5257, 0.5357, 0.6094)⟩	$U_1 > U_3 > U_2$
< 0.38, 0.34, 0.28 >	⟨(0.3851, 0.7080, 0.75360.7629), (0.3167, 0.4146, 0.5236, 0.5722), (0.3302, 0.5261, 0.5465, 0.5643)⟩ ⟨(0.4965, 0.5749, 0.5870, 0.6641), (0.5242, 0.6019, 0.6307, 0.6939), (0.2814, 0.4672, 0.5261, 0.5309)⟩ ⟨(0.4511, 0.5569, 0.5656, 0.6740), (0.4050, 0.5007, 0.5858, 0.5975), (0.4539, 0.5126, 0.5597, 0.5781)⟩	$U_3 > U_1 > U_2$

Table 2 Sensitivity analysis under L_{geo} operator

Decision makers weight	Final decision matrix	Ranking order
< 0.33, 0.37, 0.3 >	⟨(0.3744, 0.4202, 0.4539, 0.5389), (0.4633, 0.5766, 0.6512, 0.6632), (0.3374, 0.3808, 0.4830, 0.5119)⟩ ⟨(0.5138, 0.5512, 0.6269, 0.6395), (0.2051, 0.3952, 0.4612, 0.4615), (0.4459, 0.4931, 0.5589, 0.5948)⟩ ⟨(0.6565, 0.6947, 0.6998, 0.7335), (0.3370, 0.3570, 0.3698, 0.3709), (0.3425, 0.4061, 0.4238, 0.5567)⟩	$U_3 > U_2 > U_1$
< 0.36, 0.3, 0.34 >	⟨(0.2508, 0.2560, 0.2602, 0.2810), (0.4765, 0.5843, 0.6449, 1.000), (0.3315, 0.3937, 0.3992, 0.4153)⟩ ⟨(0.5134, 0.5366, 0.5396, 0.5464), (0.3853, 0.3893, 0.4357, 0.4474), (0.5426, 0.5811, 0.6303, 0.6494)⟩ ⟨(0.4463, 0.5200, 0.5217, 0.5420), (0.3956, 0.4800, 0.5529, 0.6365), (0.3683, 0.4720, 0.4898, 0.4994)⟩	$U_3 > U_2 > U_1$
< 0.33, 0.32, 0.35 >	⟨(0.2531, 0.3079, 0.3469, 0.4367), (0.4756, 0.5848, 0.6466, 1.000), (0.3319, 0.3880, 0.3899, 0.4107)⟩ ⟨(0.5060, 0.5383, 0.5393, 0.6325), (0.4007, 0.4341, 0.4369, 0.4497), (0.4376, 0.4834, 0.5397, 0.6042)⟩ ⟨(0.4519, 0.5190, 0.5202, 0.5342), (0.4580, 0.4898, 0.5377, 0.6349), (0.3605, 0.4744, 0.4854, 0.6657)⟩	$U_3 > U_2 > U_1$
< 0.28, 0.35, 0.37 >	⟨(0.6426, 0.6993, 0.7126, 0.7206), (0.5865, 0.6043, 0.6822, 1.000), (0.3608, 0.4081, 0.4084, 0.5388)⟩ ⟨(0.4776, 0.5210, 0.5960, 0.6130), (0.4296, 0.4300, 0.4900, 0.4911), (0.5083, 0.5230, 0.5962, 0.6226)⟩ ⟨(0.4295, 0.4679, 0.4697, 0.5049), (0.5218, 0.5236, 0.6036, 0.6387), (0.3695, 0.5074, 0.5603, 0.5852)⟩	$U_3 > U_2 > U_1$
< 0.38, 0.34, 0.28 >	⟨(0.6689, 0.7134, 0.7576, 0.7676), (0.5671, 0.5801, 0.6576, 1.000), (0.4451, 0.4978, 0.5046, 0.5298)⟩ ⟨(0.5223, 0.5439, 0.6298, 0.6430), (0.3826, 0.3945, 0.4654, 0.4675), (0.2662, 0.2997, 0.3545, 0.4101)⟩ ⟨(0.6389, 0.6805, 0.7089, 0.7422), (0.3947, 0.4736, 0.4770, 0.4881), (0.3593, 0.4800, 0.5179, 0.5709)⟩	$U_3 > U_2 > U_1$

$$\left(\mathbf{C^1_{3\times1}}\right)_{\mathbf{geo}} = \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix} \begin{pmatrix} \langle(0.6082, 0.6306, 0.6357, 0.6407), (0.5090, 0.5215, 0.5524, 0.7129), (0.7560, 0.7792, 0.8012, 0.8669)\rangle \\ \langle(0.4192, 0.4513, 0.4580, 0.4875), (0.1874, 0.3897, 0.4195, 0.5780), (0.6056, 0.6210, 0.7190, 0.7517)\rangle \\ \langle(0.3274, 0.3362, 0.6174, 0.6395), (0.4056, 0.4459, 0.5245, 0.6967), (0.6333, 0.6353, 0.6459, 0.6949)\rangle \end{pmatrix},$$

$$\left(\mathbf{C^2_{3\times1}}\right)_{\mathbf{geo}} = \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix} \begin{pmatrix} \langle(0.3225, 0.4421, 0.4786, 0.4830), (0.5079, 0.5114, 0.5963, 1.000), (0.2675, 0.2694, 0.3819, 0.4608)\rangle \\ \langle(0.5030, 0.5648, 0.6040, 0.6594), (0.5438, 0.5485, 0.5697, 0.5707), (0.5411, 0.5915, 0.7290, 0.7745)\rangle \\ \langle(0.4516, 0.4594, 0.5232, 0.5427), (0.5773, 0.5573, 0.6377, 0.6603), (0.1731, 0.3048, 0.3856, 0.4461)\rangle \end{pmatrix},$$

$$\left(\mathbf{C^3_{3\times1}}\right)_{\mathbf{geo}} = \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix} \begin{pmatrix} \langle(0.4384, 0.4432, 0.4583, 0.4710), (.5632, 0.6157, 0.6212, 0.6592), (0.2922, 0.3089, 0.3482, 0.3686)\rangle \\ \langle(0.3808, 0.4842, 0.5205, 0.548), (0.4578, 0.5433, 0.5562, 0.5752), (0.3162, 0.3162, 0.3705, 0.3862)\rangle \\ \langle(0.7554, 0.7998, 0.8050, 0.8111), (0.4014, 0.5954, 0.6002, 0.6181), (0.4040, 0.4893, 0.5954, 0.6213)\rangle \end{pmatrix}.$$

Step 2: We now apply decision-maker's weight maintaining the relation $\sum_{k=1}^{w} \Omega_k C^k_{u\times1}$ (scalar multiplication and addition of TNNs) and we have overall attribute values $\widetilde{A}_{s1}$ for the alternatives U_s ($s = 1, 2, 3$) under the operator L_{arm} as follows

$$\mathbf{(DM)_{arm}} = \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix} \begin{pmatrix} \langle(0.3836, 0.7019, 0.7575, 0.7615), (0.3193, 0.4179, 0.5298, 0.6054), (0.3380, 0.4476, 0.5765, 0.5886)\rangle \\ \langle(0.5044, 0.5680, 0.5910, 0.6677), (0.5063, 0.6009, 0.6350, 0.6995), (0.3069, 0.3727, 0.5138, 0.5494)\rangle \\ \langle(0.4278, 0.6474, 0.6789, 0.6819), (0.4059, 0.4902, 0.5858, 0.6171), (0.4731, 0.5164, 0.5533, 0.5951)\rangle \end{pmatrix}.$$

On the other side, if we apply decision-makers weight under the operator L_{geo} according to the relation $\sum_{k=1}^{w} \Omega_k C^k_{u\times1}$, we get overall attribute values $\widetilde{A}_{s1}$ for the alternatives U_s ($s = 1, 2, 3$) which is given as

$$\mathbf{(DM)_{geo}} = \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix} \begin{pmatrix} \langle(0.3744, 0.4202, 0.4539, 0.5389), (0.4633, 0.5766, 0.6512, 0.6632), (0.3374, 0.3808, 0.4830, 0.5119)\rangle \\ \langle(0.5138, 0.5512, 0.6269, 0.6395), (0.2051, 0.3952, 0.4612, 0.4615), (0.4459, 0.4931, 0.5589, 0.5948)\rangle \\ \langle(0.6565, 0.6947, 0.6998, 0.7335), (0.337, 0.3570, 0.3698, 0.3709), (0.3425, 0.4061, 0.4238, 0.4567)\rangle \end{pmatrix}.$$

Step 3: The de-Neutrosofication values of $\widetilde{A}_{s1}$, ($s = 1,2,3$) corresponding to L_{arm} operator are $D_{Neu}(\widetilde{A}_{11}) = 0.5208$, $D_{Neu}(\widetilde{A}_{21}) = 0.4882$, $D_{Neu}(\widetilde{A}_{31}) = 0.5837$. On the other hand, the de-Neutrosofication values of $\widetilde{A}_{s1}$, ($s = 1,2,3$) corresponding to operator L_{geo} are $D_{Neu}(\widetilde{A}_{11}) = 0.5156$, $D_{Neu}(\widetilde{A}_{21}) = 0.51$, $D_{Neu}(\widetilde{A}_{31}) = 0.5601$.

Step 4: The ranking order of de-Neutrosofication values is $D_{Neu}(\widetilde{A}_{31}) > D_{Neu}(\widetilde{A}_{11}) > D_{Neu}(\widetilde{A}_{21})$ for the operator L_{arm}. Therefore, the ranking order of the alternatives is $U_3 > U_1 > U_2$. Therefore, $U3$ is the best option. Again, under the operator L_{geo}, the ranking order of the alternatives is $U_3 > U_2 > U_1$. Therefore, U_3 is the best option.

6.1 Sensitivity analysis

The logical approach of sensitivity analysis is performed by exchanging the weights of the decision-makers keeping the remainder of the term are unchanged. Here, we perform sensitivity analysis under the L_{arm} and L_{geo} operators to capture the influence of the decision-makers weight on the relative matrix and their ranking. The sensitivity analysis results are shown in the Tables 1 & 2 under the operators L_{arm} and L_{geo} respectively. In the Figs. 3 and 4, we have represented the corresponding weights values of different decision-makers and the ranking order of the alternatives respectively under L_{arm} operator. Also, in the Figs. 5 and 6, we have presented the related weights values of different decision-makers and the ranking order of the alternatives respectively under L_{geo} operator.

In Table 1, we consider different weight vectors of the decision-makers and get U_3 is the best option in four cases and U_1 is the best option for one case under the operator L_{arm}. Again, in Table 2, we consider same weight vectors of the decision-makers as in Table 1 and get U_3 is the best option in all cases under the operator L_{geo}.

6.2 Comparative analysis

To demonstrate the efficiency and validity of our proposed method, we have presented a comparison study of our method with the existing methods in Table 3.

From the Table 3, we have observed that aggregation operator proposed by Ye [15] cannot be apply in our decision matrices as indeterminacy part is absent in this

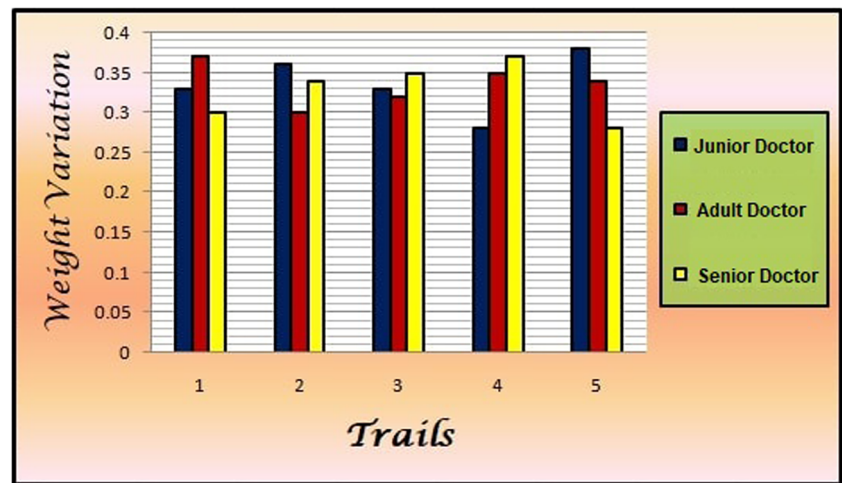**Fig. 3** Different decision-maker's weights under the operator L_{arm}

Fig. 4 Ranking order of the alternatives under the operator L_{arm}
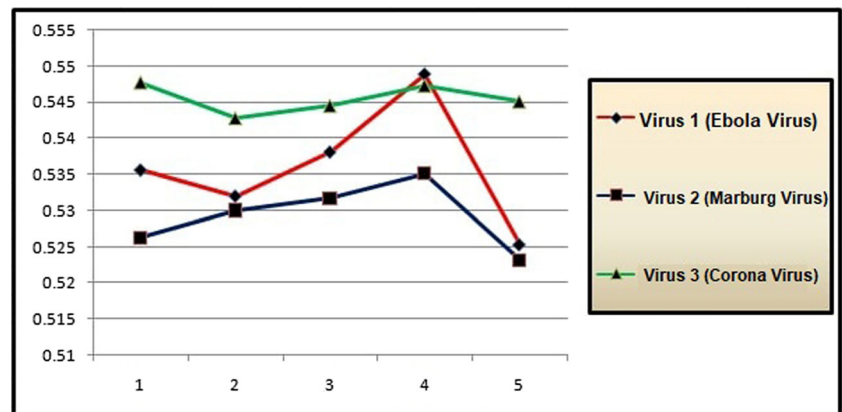

Fig. 5 Different decision-maker's weights under the operator L_{geo}
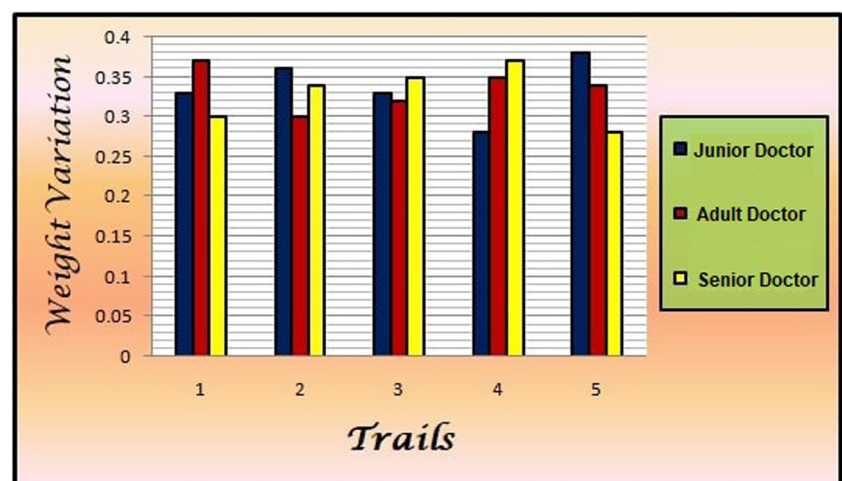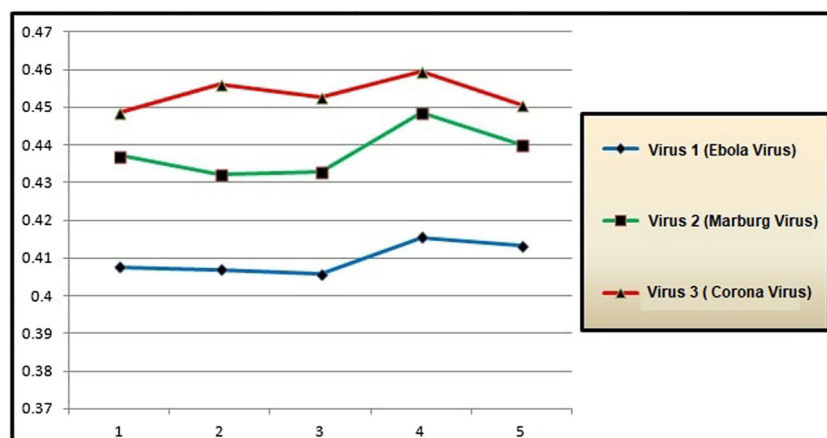
Fig. 6 Ranking order of the alternatives under the operator L_{geo}

aggregation operator. Also Liang et al. [20], Biswas et al. [23], Pranab et al. [34], Pramanik & Mallick [35], Liu & Zhang [36] and Wu et al. [37] work on SVTNN environment which is different from general TNN [16] through its basic character. So we cannot apply this method in our decision matrices to execute the best alternative. Thus, we have applied the operators TNNWAA [16], TNNWGA [16], ITNNWAA [17] and ITNNWGA [17] on our data set and obtained the results. Interestingly, we have found that the ranking order under the different operators and our method are exactly same. On the other hand we have already checked the stability of our obtained results through sensitivity analysis. These phenomenons clearly show the efficiency & reliability of our proposed logarithmic operational law based MCGDM technique.

7 Conclusion

In this article, we have presented new logarithmic operational laws for TNNs which is a productive enhancement of existing operational laws. We have studied their mathematical Properties like boundedness, monotonicity etc. Moreover, we have proposed the logarithmic trapezoidal neutrosophic weighted arithmetic aggregation operator L_{arm} and logarithmic trapezoidal neutrosophic weighted geometric aggregation operator L_{geo} and presented an MCGDM technique in TN environment by using these aggregation operators. A numerical problem has been taken up to demonstrate the proposed MCGDM method. Also, we have discussed the usefulness and the utility of the proposed method through a sensitivity analysis. Finally, a comparison study of our proposed technique with existing methods has been presented to justify the rationality and efficiency of our proposed technique. From this article, we can conclude that our defined operational law and its corresponding MCGDM technique give a new direction to deal decision-making problems.

In the future work, the defined logarithmic operational law can be expanded to the other uncertain environments to enrich the decision-making procedure. Researchers can immensely apply these ideas of neutrosophic number in numerous flourishing research fields like mobile computing, pattern recognition, cloud computing, etc.

Table 3 Comparison with the existing methods

| Methods	Nature of the environment	Multiple group of decision makers	Operators	Ranking order
Ye [15]	TrIFN	×	TIFPWA TIFPWG	Not applicable
Liang et al. [20]	SVTNN	×	SVTNWAA SVTNWGA	Not applicable
Biswas et al. [23]	SVTNN	×	×	Not applicable
Pramanik & Mallick [35]	SVTNN	✓	×	Not applicable
Liu & Zhang [36]	SVTNN	✓	SVTNWMSM	Not applicable
Wu et al. [37]	SVTNN	✓	SVTNPA SVTNPG	Not applicable
Ye [16]	TNN	×	TNNWAA TNNWGA	$U_3 > U_2 > U_1$ $U_3 > U_2 > U_1$
Jana et al. [17]	TNN	×	ITNNWAA ITNNWGA	$U_3 > U_2 > U_1$ $U_3 > U_2 > U_1$
Proposed Method	TNN	✓	L_{arm} L_{geo}	$U_3 > U_2 > U_1$ $U_3 > U_2 > U_1$

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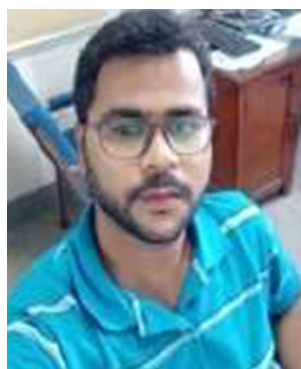
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