A SET OF NEW SMARANDACHE FUNCTIONS, SEQUENCES AND CONJECTURES IN NUMBER THEORY

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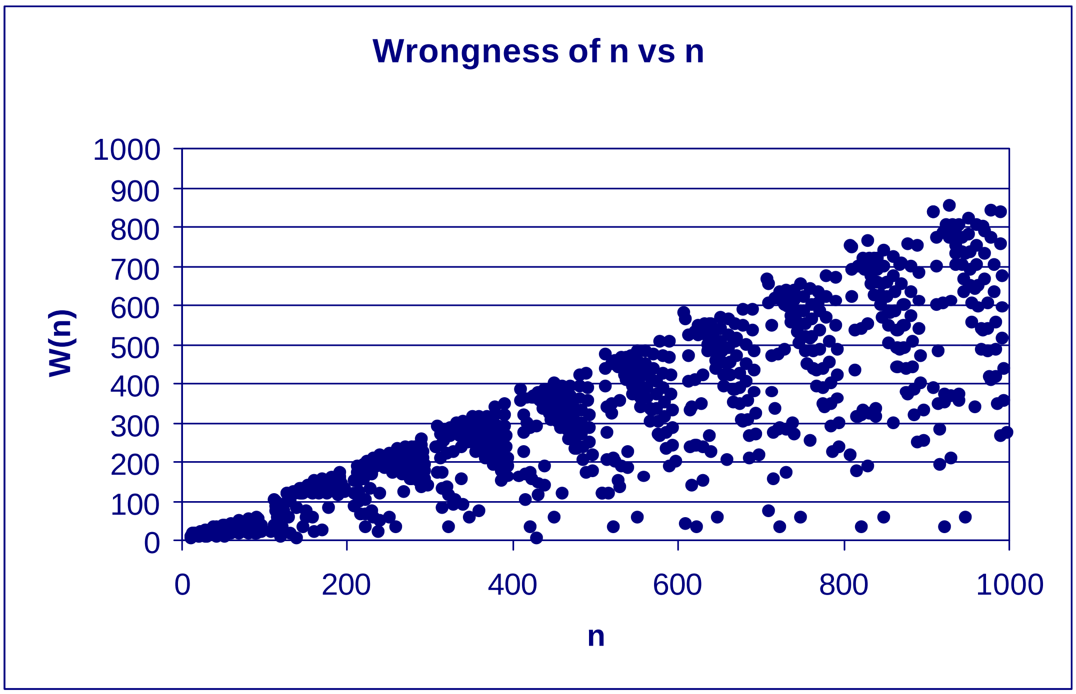

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To two stars:
my doughters ***Gilda Aldebaran*** and ***Francesca Carlotta Antares***

To the memory of my brother ***Felice*** and my late uncles ***Raffaele*** and ***Angelo*** and my late ant ***Jolanda*** who surely would have appreciated this work of mine.

INTRODUCTION

I have met the Smarandache's world for the first time about one year ago reading some articles and problems published in the Journal of Recreational Mathematics.

From then on I discovered the interesting American Research Press web site dedicated to the Smarandache notions and held by Dr. Perez (address: http://www.gallup.unm.edu/~smarandache/), the Smarandache Notions Journal always published by American Research Press, and several books on conjectures, functions, unsolved problems, notions and other proposed by Professor F. Smarandache in "The Florentin Smarandache papers" special collections at: the Arizona State University (Tempe, USA), Archives of American Mathematics (University of Texas at Austin, USA), University of Craiova Library (Romania), and Archives of State (Rm. Valcea, Romania).

The Smarandache's universe is undoubtedly very fascinating and is halfway between the number theory and the recreational mathematics.
Even though sometime this universe has a very simple structure from number theory standpoint, it doesn't cease to be deeply mysterious and interesting.

This book, following the Smarandache spirit, presents new Smarandache functions, new conjectures, solved/unsolved problems, new Smarandache type sequences and new Smarandache Notions in number theory.
Moreover a chapter (IV) is dedicated to the analysis of Smarandache Double factorial function introduced in number theory by F. Smarandache ("The Florentin Smarandache papers" special collection, University of Craiova Library, and Archivele Statului, Filiala Valcea) and another one (V) to the study of some conjectures and open questions proposed always by F. Smarandache.
In particular we will analyse some conjectures on prime numbers and the generalizations of Goldbach conjecture.
This book would be a telescope to explore and enlarge our knowledge on the Smarandache's universe. So let's start our observation.

Chapter I

On some new Smarandache functions in Number Theory.

A number-theoretic function is any function which is defined for positive integers argument.

Euler's function $\varphi(n)$ [3] is such, as are $n!$, $\lfloor n \rfloor$, n^2 etc. The functions which are interesting from number theory point of view are, of course, those like $\varphi(n)$ whose value depends in some way on the arithmetic nature of the argument, and not simply on its size. But the behaviour of the function is likely to be highly irregular, and it may be a difficult matter to describe how rapidly the function value grows as the argument increases. In the 1970's F. Smarandache created a new function in number theory whose behaviour is highly irregular like the $\varphi(n)$ function.

Called the Smarandache function in his honor it also has a simple definition:

if n>0, then S(n)=m, where m is the smallest number ≥ 0 *such that n evenly divides m! [1]*

In the 1996, K. Kashihara [2] defined, analogously to the Smarandache function, the Pseudo Smarandache function:

given any integer $n \geq 1$*, the value of the Pseudo Smarandache function Z(n), is the smallest integer m such that n evenly divides the sum of first m integers.*

In this chapter we will define four other Pseudo Smarandache functions in number theory analogous to the Pseudo-Smarandache function.

Many of the results obtained for these functions are similar to those of Smarandache and Pseudo-Smarandache functions. Several examples, conjectures and problem are given too. Regarding some proposed problems a partial solution is sketched.

1.1 PSEDUO-SMARANDACHE-TOTIENT FUNCTION

The Pseudo Smarandache totient function Zt(n) is defined as the smallest integer m such that:

$$\sum_{k=1}^{m} j\,(k)$$

is divisible by n. Here $j\,(n)$ is the Euler (or totient) function that is the number of positive integers $k \leq n$ which are relatively prime to n [3].
In the figure 1.1, the growth of function Zt(n) versus n is showed. As for the Euler function its behaviour is highly erratic.

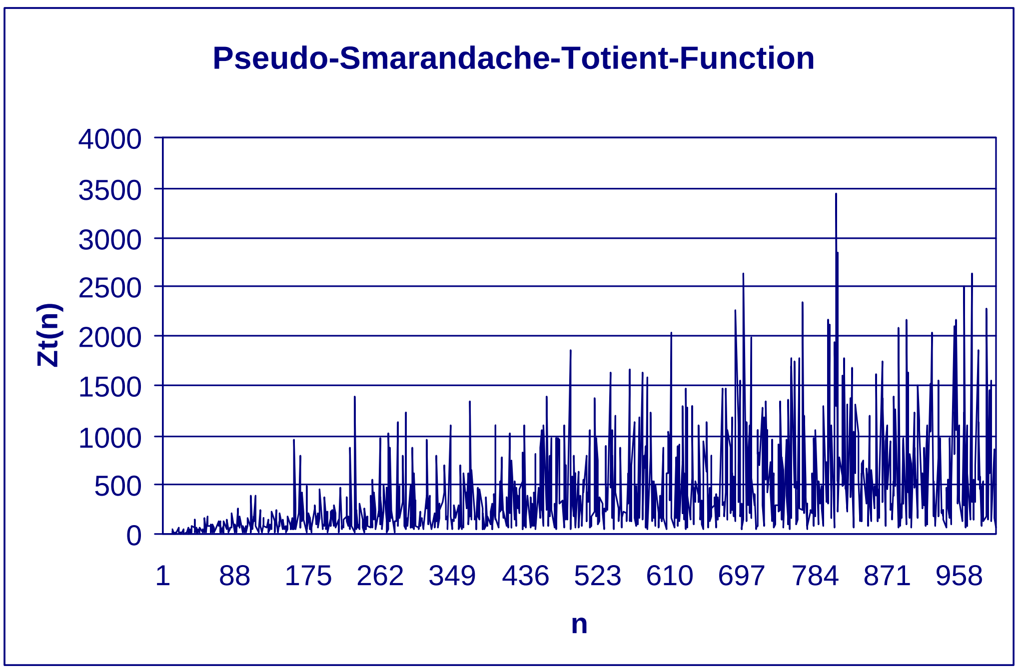

Fig. 1.1

Anyway using a logarithmic y axis a clear pattern emerges. We can see that the points of Zt function tend to dispose along curves that grow like the square root of n. (Fig. 1.2) In fact according to Walfisz result [3] the sum of first m values of Euler function is given by:

$$\sum_{k=1}^{m} \varphi(k) = \frac{3 \cdot m^2}{\pi^2} + O(m \cdot \ln(m)^{\frac{2}{3}} \cdot (\ln(\ln(m)))^{\frac{4}{3}}$$

and then the Zt(n) asymptotic behaviour is decribed as :

$$Zt(n) = m \approx \pi \cdot \sqrt{\frac{k \cdot n}{3}} \qquad \text{for } k \in N$$

where the k parameter modulates Zt(n).

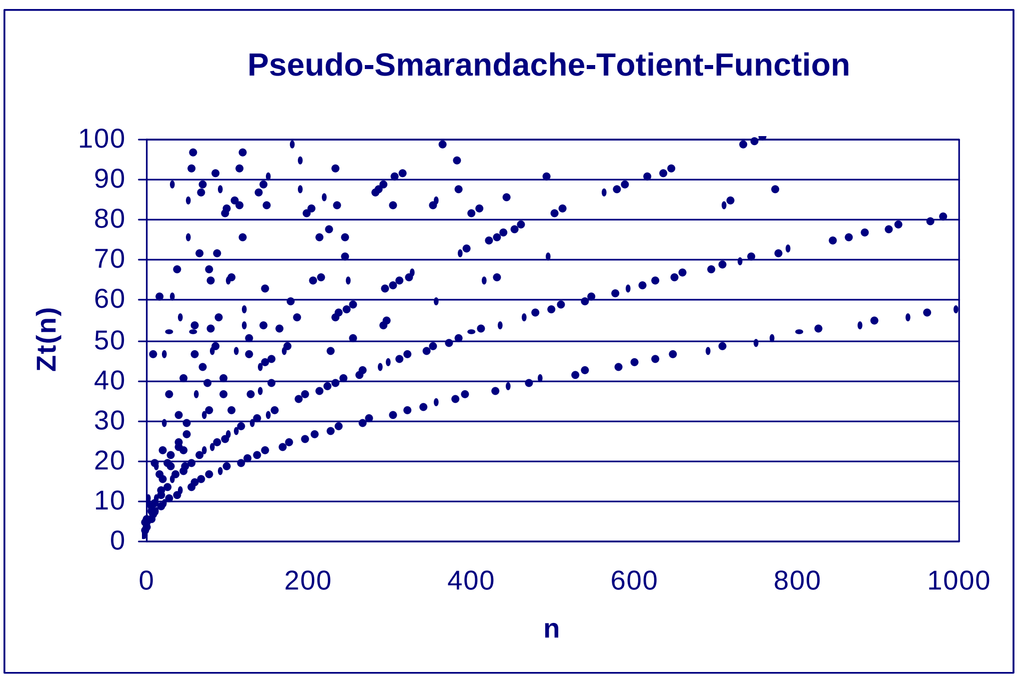

Fig. 1.2

A table of the Zt(n) values for $1 \le n \le 60$ follows:

n	Zt(n)	n	Zt(n)	n	Zt(n)
1	1	21	11	41	67
2	2	22	8	42	11
3	4	23	12	43	23
4	3	24	15	44	31
5	5	25	22	45	24
6	4	26	46	46	12
7	9	27	29	47	55
8	10	28	9	48	17
9	7	29	13	49	40
10	5	30	19	50	22
11	8	31	51	51	18
12	6	32	10	52	153
13	46	33	36	53	26
14	9	34	18	54	29
15	19	35	21	55	184
16	10	36	15	56	75
17	18	37	88	57	84
18	7	38	60	58	13
19	60	39	142	59	92
20	16	40	16	60	19

Let's start now to explore some properties of this new function.

Theorem 1.1.1 *The Zt(n) function is not additive and not multiplicative, that is* $Zt(m \cdot n) \neq Zt(m) \cdot Zt(n)$ *and* $Zt(m+n) \neq Zt(m) + Zt(n)$ *[8].*

Proof. In fact for example: $\mathrm{Zt}(2+3) \neq \mathrm{Zt}(2) + \mathrm{Zt}(3)$ and $\mathrm{Zt}(2 \cdot 3) \neq \mathrm{Zt}(2) \cdot \mathrm{Zt}(3)$

Theorem 1.1.2 *Zt(n)>1 for n >1*

Proof. This is due to the fact that $j\,(\mathrm{n}) > 0$ for n>0 and $j\,(\mathrm{n}) = 1$ only for n=1.
Note that Zt(n)=1 if and only if n=1.

Theorem 1.1.3 $\sum_{k=1}^{Zt(n)} j(k) \leq \frac{Zt(n)\cdot(Zt(n)+1)}{2} \qquad for\ \ n \geq 1$

Proof. Assume Zt(n)=m. Since $j(n) \leq n$ for $n \geq 1$ this implies that

$$\sum_{k=1}^{m} j(k) \leq \sum_{k=1}^{m} k = \frac{m\cdot(m+1)}{2}$$

Theorem 1.1.4 $\sum_{n=1}^{\infty} \frac{1}{Zt(n)}$ *diverges*.

Proof. By definition Zt(n)=m, and this implies that $\sum_{k=1}^{m} j(k) = a\cdot n$ where $a \in N$.

Then $\frac{3\cdot m^2}{p^2} \approx a\cdot n$ according to Walfisz result reported previously [3].

Therefore $m \approx \sqrt{\frac{p^2\cdot a\cdot n}{3}}$ *and* $\sum_{n=1}^{\infty} \frac{1}{Zt(n)} \approx \sum_{n=1}^{\infty} \frac{1}{p\cdot\sqrt{\frac{a\cdot n}{3}}} > \frac{3}{a\cdot p}\sum_{n=1}^{\infty} \frac{1}{n}$

diverges, because as known: $\lim_{n\to\infty} \sum_{n} \frac{1}{n} \to \infty$

Conjecture 1.1.1 The sum of reciprocals of Zt(n) function is asymptotically equal to the natural logarithm of n:

$$\sum_{n=1}^{N} \frac{1}{Zt(n)} \approx a\cdot\ln(N) + b \quad \text{where} \quad a \approx 0.9743\ldots \quad \text{and} \quad b \approx 0.739\ldots$$

Theorem 1.1.5 $\sum_{n=1}^{\infty} \frac{Zt(n)}{n}$ *diverges*

Proof. In fact:

$$\sum_{n=1}^{\infty} \frac{Zt(n)}{n} \approx p \cdot \sum_{n=1}^{\infty} \frac{\sqrt{\frac{a \cdot n}{3}}}{n} > \sum_{n=1}^{\infty} \frac{1}{n}$$

and as known the sum of reciprocals of natural numbers diverges.

Conjecture 1.1.2 $\sum_{k=1}^{N} \frac{Zt(k)}{k} \approx a \cdot N \quad where \; a \approx 0.8737K$

Theorem 1.1.6 $n \le \frac{p^2}{3} \cdot \sum_{k=1}^{n} j(k)$

Proof. According to Walfisz result $\sum_{k=1}^{n} j(k) \approx \frac{3 \cdot n^2}{p^2}$, and then the theorem is a consequence of inequality $n \le n^2$.

Theorem 1.1.7 $\sum_{k=1}^{Zt(n)} j(k) \ge n$

Proof. The result is a direct consequence of the Zt(n) definition. In fact

$$\sum_{k=1}^{m} j(k) = a \cdot n \quad where \; a \in N \; .$$

For a=1 we have $\sum_{k=1}^{m} j(k) = n$ while for a>1 $\sum_{k=1}^{m} j(k) > n$

Theorem 1.1.8 $Zt(n) \ge \left\lfloor p \cdot \sqrt{\frac{n}{3}} \right\rfloor$

Proof. The result is a consequence of definition of Zt(n). In fact:

$$Zt(n) \approx p \cdot \sqrt{\frac{a \cdot n}{3}} \geq \left\lfloor p \cdot \sqrt{\frac{n}{3}} \right\rfloor$$

where $a \in N$ and the symbol $\lfloor n \rfloor$ indicated the floor function [3], that by definition is the largest integer $\leq n$. In many computer languages, the floor function is called the integer part function and is denoted int(n).

Theorem 1.1.9 *It is not always the case that Zt(n)<n*

Proof. Examine for example the following values of Zt(n): Zt(3)=4, Zt(7)=9 and so on.

Theorem 1.1.10 *The range of Zt(n) function is N-{0} where N is the set of positive integers numbers.*

Proof. The theorem is a direct consequence of Walfistz result [3]. In fact for each number m we can found a number n given approximatively by:

$$n \approx \frac{3 \cdot m^2}{a \cdot p^2} \qquad where \quad a \in N$$

such that Zt(n)=m.

As for the Smarandache and Pseudo Smarandache function, there are several open problems involving the Pseudo Smarandache Totient function. Some of those problems are presented herebelow.

Problem 1. *Find the largest number k such that Zt(n), Zt(n+1), Zt(n+2).....Zt(n+k) are all increasing (decreasing respectively) values.*
For the first 1000 values of Zt(n) the largest found sequences have k=5 and k=4 respectively.

Example:

Zt(514) < Zt(515) < Zt(516) < Zt(517) < Zt(518) < Zt(519)
Zt(544) > Zt(545) > Zt(546) > Zt(547) > Zt(548)

Conjecture 1.1.3 The parameter k is upper limited.

Unsolved question: Find that upper limit.

Problem 2. *Find the solution of Zt(n)=n*

For the first 1000 terms only for n=1, 2 and 5 the equation is satisfied. Are the solutions unique ?
In this case we must solve the following equation:

$$\sum_{k=1}^{n} j(k) = a \cdot n \quad \text{where } a \in N$$

Problem 3. *Let's indicate with A how many times Zt(n)<n and with B how many times Zt(n)>n. Evaluate* $\lim_{n\to\infty} \frac{A}{B}$

Conjecture 1.1.4 The limit is finite and greater than 1.

Problem 4. *Are the following values bounded or unbounded?*

$$d_n = |Zt(n+1) - Zt(n)|$$

$$r_n = \frac{Zt(n+1)}{Zt(n)}$$

$$L_n = \frac{|Zt(n) - Zt(m)|}{|n-m|} \quad \text{where } n, m \in N$$

Hint: the experimental data on the first 1000 values of Zt(n) show a linear grows for the average value of d_n.
So this should implies that d_n is unbounded.
About r_n the experimental data show no clear trend. So any conclusion is difficult to drive.

Problem 5. *Find all values of n such that:*

1) Zt(n)|Zt(n+1)
2) Zt(n+1)|Zt(n)

Examining the first 1000 values of Zt(n), the following solutions to 1) have been found:

Zt(1)|Zt(2), Zt(2)|Zt(3), Zt(80)|Zt(81), Zt(144)|Zt(145), Zt(150)|Zt(151), Zt(396)|Zt(397), Zt(549)|Zt(550), Zt(571)|Zt(572), Zt(830)|Zt(831)

Unsolved question*:* Is the number of those solutions limited or unlimited?

Solutions to 2) for the first 1000 values of Zt(n):

Zt(34)|Zt(33), Zt(46)|Zt(45), Zt(75)|Zt(74), Zt(86)|Zt(85),
Zt(90)|Zt(89), Zt(108)|Zt(107), Zt(172)|Zt(171), Zt(225)|Zt(224),
Zt(242)|Zt(241), Zt(464)|Zt(465), Zt(650)|Zt(649), Zt(886)|Zt(885)

Unsolved question: Is the number of those solutions limited or unlimited?

If we indicate with C the number of solutions to 1) and with D the solutions to 2), evaluate:

$$\lim_{n\to\infty} \frac{D}{C}$$

$$\lim_{n\to\infty} \frac{(D-C)^2}{\left|C^2-D^2\right|}$$

Evaluate the number of times C and D are equal and the value of n in correspondence of which the difference is zero.

Problem 6. *Find all values of n such that Zt(n+1)=Zt(n).*

Conjecture 1.1.5 There are no solutions to the Diophantine equation Zt(n+1)=Zt(n).

Experimental data support this conjecture.
Infact the behaviour of d_n (see problem 4 for definition) looks like to point-out that never $d_n = 0$.

Problem 7*.* *Is there any relationship between:*

- *Zt(n+m) and Zt(n), Zt(m)*
- $Zt(n \cdot m)$ *and Zt(N), Zt(m) ?*

Problem 8*.* *Let's consider the functions Zt(n) and* $j(n)$*. If we indicate with K how many times* $Zt(n) > j(n)$ *and with L how many times* $Zt(n) < j(n)$ *evaluate the ratio:*

$$\lim_{n\to\infty} \frac{K}{L}$$

Examining the table of values of Zt(n) and $j(n)$ for the first 100 values of n, we have that $Zt(n) > j(n)$ for:

n= 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24, 25, 26, 27......

while $Zt(n) < j(n)$ for:

n=11, 21, 22, 23, 28, 29, 32, 35, 42, 43, 46, 49, 51.......

For $1 \le n \le 10000$ the equation $Zt(n) = j(n)$ admits the following 9 solutions:

n=1, n=40, n=45, n=90, n=607, n=1025, n=1214, n=2050, n=5345

Unsolved question:

- Is the number of solutions of equation $Zt(n) = j(n)$ upper limited?
- Evaluate how many times |K-L|=0

Problem 9*.* *Analyze the iteration of Zt(n) for all values of n. For iteration we intend the repeated application of Zt(n).*

For example the k-th iteration of Zt(n) is:

$$Zt^{k}(n) = Zt(Zt(Zt(\mathrm{K}\ (Zt(n))\mathrm{K}\ ) \qquad \text{where Zt is repeated k times.}$$

Unsolved question: For all values of n, will always each iteration of Zt(n) produce a fixed point or a cycle?

A fixed point by definition [3] is a point which does not change upon repeated application of the Zt function.

An n-cycle, instead, is a finite sequence of points $Y_0, Y_1, \mathrm{K}\ Y_{n-1}$ such that, under the function Zt,

$$Y_1 = Zt(Y_0)$$
$$Y_2 = Zt(Y_1)$$
$$Y_{n-1} = Zt(Y_{n-2})$$
$$Y_0 = Zt(Y_{n-1})\,.$$

In other words, it is a periodic trajectory which comes back to the same point after n iterations of the cycle. A fixed point is a cycle of period 1. [3]
Example of fixed point . For n=234 we have the following result

$$234 \to 241 \to 56 \to 75 \to 22 \to 8 \to 5 \to 5 \to 5 \ldots$$

Example of 2-Cycle. For n=154 we have:

$$154 \to 31 \to 51 \to 18 \to 7 \to 9 \to 7 \to 9 \ldots$$

Problem 10. *If every integer n produces a cycle or a fixed point, which is the cycle with the largest period ?*

Problem 11. *Solve the equation: Zt(n)+Zt(n+1)=Zt(n+2)*
For the first 1000 values of Zt(n), one solution has been found: Zt(6)+Zt(7)=Zt(8).

Unsolved question: Is the number of solutions finite?

Problem 12. *Solve the equation: Zt(n)=Zt(n+1)+Zt(n+2)*
For the first 1000 values of Zt(n) again one solution has been found: Zt(49)=Zt(50)+Zt(51)

Unsolved question: How many other solutions do exist?

Problem 13. *Solve the equation:* $Zt(n) = Zt(n+1) \cdot Zt(n+2)$
No solution has been found for the first 1000 values of Zt(n).

Unsolved question: Is it this true for all n?

Examining the results of a computer search looks like that the following inequality hold: $Zt(n) < Zt(n+1) \cdot Zt(n+2)$. Is this true for all values of n? If yes prove why.

Problem 14. *Find all values of n such that* $Zt(n) \cdot Zt(n+1) = Zt(n+2)$
Also in this case no solution have been found for the first 1000 values of Zt(n).

Unsolved question: Is this true for all values of n?

Checking the inequality $Zt(n) \cdot Zt(n+1) > Zt(n+2)$ only one solution among the first 1000 values of Zt(n) has been found: $Zt(1) \cdot Zt(2) > Zt(3)$. Is that solution unique for all values of n?

Problem 15. *Find all values of n such that* $Zt(n) \cdot Zt(n+1) = Zt(n+2) \cdot Zt(n+3)$.
For the first 1000 values of Zt(n) no solution has been found. Is it this true for all n?
If yes, why?

Problem 16. *Solve the equation Z(n)=Zt(n), where Z(n) is the Pseudo-Smarandache function [2].*
For the first 60 values of Zt(n) and Z(n) two solutions have been found: Z(1)=Zt(1), Z(24)=Zt(24)=15

Unsolved question: How many other solutions do exist for all values of n?

Problem 17. *Find all values of n such that: Zt(n)=Z(n) +/- 1*
For the first 60 values of Zt(n) and Z(n) the following solutions have been found:

Zt(2)=Z(2)-1	Zt(5)=Z(5)+1
Zt(9)=Z(9)-1	Zt(6)=Z(6)+1
Zt(18)=Z(18)-1	Zt(10)=Z(10)+1
Zt(44)=Z(44)-1	Zt(20)=Z(20)+1
	Zt(40)=Z(40)+1
	Zt(51)=Z(51)+1

Unsolved question: Is the number of solutions upper limited?

Problem 18. *Solve the equation S(n)=Zt(n) where S(n) is the Smarandache function [1].*
For the first 84 values of S(n) four solutions have been found:

S(1)=Zt(1)=1, S(2)=Zt(2)=2, S(5)=Zt(5)=5, S(10)=Zt(10)=5

Unsolved question: How many other solutions do exist for all values of n?

For those first values note that $S(n) = Zt(n) = n \ or \ S(p \cdot q) = Zt(p \cdot q) = q$ where p and q are two distinct primes and q>p.
Is it true for all the solutions of the equation S(n)=Zt(n)? If yes why?

Problem 19. *Find all values of n such that S(n)=Zt(n) +/- 1*
For the first 84 values of S(n) and Zt(n) the following solutions have been found:

S(4)=Zt(4)+1	S(3)=Zt(3)-1
	S(6)=Zt(6)-1
	S(9)=Zt(9)-1
	S(17)=Zt(17)-1
	S(18)=Zt(18)-1
	S(34)=Zt(34)-1
	S(51)=Zt(51)-1

Unsolved question: Is the number of solutions upper limited?

If we look to the solutions of equation S(n)=Zt(n)-1, we can see that we have as solutions two consecutive integers: 17 and 18.
How many other consecutive integers are solutions of this equation?
Which is the maximum number of consecutive integers?

Problem 20. *Find all values of n such that* $S(n) = 2 \cdot Zt(n) - Z(n)$
For $1 \leq n \leq 84$ the following two solutions have been found:

$$S(9) = 2 \cdot Zt(9) - Z(9), \qquad S(18) = 2 \cdot Zt(18) - Z(18)$$

Is there any relationship among the solutions of this equation?

Problem 21. *Solve the equation Zt(p)=p' where p and p' are different primes.*
For the first 60 values of Zt(n) three solutions have been found:

Zt(29)=13, Zt(41)=67, Zt(43)=23

Unsolved question: How many other solutions do exist?

Problem 22. *Solve the equation: Zt(p)=p where p is any prime.*
For the first 60 values of Zt(n) two solutions have been found:

Zt(2)=2 and Zt(5)=5

Unsolved question: Are those the only possible solutions?

Problem 23. *Find the smallest k such that between Zt(n) and Zt(k+n), for n>1, there is at least a prime.*

Problem 24. *Solve the equation Zt(Z(n))-Z(Zt(n))=0 .*

Problem 25. *Find all values of n such that Zt(Z(n))-Z(Zt(n))>0*

Problem 26. *Find all values of n such that Zt(Z(n))-Z(Zt(n))<0*

Problem 27. *Study the functions Zt(Z(n)), Z(Zt(n)) and Zt(Z(n))-Z(Zt(n)).*

Problem 28. *Evaluate* $\lim\limits_{n\to\infty}\frac{Z_1}{Z_2}$ *where* $Z_1=\sum\limits_n Zt(Z(n))$ *and* $Z_2=\sum\limits_n Z(Zt(n))$

Problem 29. *Evaluate*

$$\lim_{n\to\infty}\frac{\sum\limits_n \left|Zt(Z(n))-Z(Zt(n))\right|}{\left|\sum\limits_n Zt(Z(n))-\sum\limits_n Z(Zt(n))\right|}$$

Problem 30. *Evaluate*

$$\lim_{n\to\infty}\frac{\left(\sum\limits_n \left|Zt(Z(n))-Z(Zt(n))\right|\right)^2}{\sum\limits_n \left(Zt(Z(n))-Z(Zt(n))\right)^2}$$

Problem 31. *Evaluate*

$$\lim_{n\to\infty}\left|\sum_n \frac{1}{Zt(Z(n))}-\sum_n \frac{1}{Z(Zt(n))}\right|$$

Problem 32. *Evaluate*

$$\lim_{n\to\infty}\sum_n \frac{Zt(n)}{Z(n)}$$

Problem 33. *Study the function F(n)=S(Z(Zt(n)))*

Problem 34. *Evaluate*

$$\lim_{n\to\infty}\sum_n \left|Zt(Z(n)) - Z(Zt(n))\right|$$

Problem 35. *As for the Smarandache function the following Smarandache sums can be defined for Z(n) and Zt(n)*

$$Zt_1 = \sum_n \frac{1}{(Zt(n))!} \qquad Z_1 = \sum_n \frac{1}{(Z(n))!}$$

$$Zt_2 = \sum_n \frac{Zt(n)}{n!} \qquad Z_2 = \sum_n \frac{Z(n)}{n!}$$

$$Zt_3 = \sum_n \frac{1}{\prod_{i=1}^{n} Zt(i)} \qquad Z_3 = \sum_n \frac{1}{\prod_{i=1}^{n} Z(i)}$$

$$Zt_4(a) = \sum_n \frac{n^a}{\prod_{i=1}^{n} Zt(i)} \qquad Z_4(a) = \sum_n \frac{n^a}{\prod_{i=1}^{n} Z(i)}$$

$$Zt_5 = \sum_n \frac{(-1)^{n-1}\cdot Zt(n)}{n!} \qquad Z_5 = \sum_n \frac{(-1)^{n-1}\cdot Z(n)}{n!}$$

$$Zt_6 = \sum_n \frac{Zt(n)}{(n+1)!} \qquad Z_6 = \sum_n \frac{Z(n)}{(n+1)!}$$

$$Zt_7 = \sum_{n=r}^{\infty} \frac{Zt(n)}{(n+r)!} \qquad Z_7 = \sum_{n=r}^{\infty} \frac{Z(n)}{(n+r)!}$$

$$Zt_8 = \sum_{n=r}^{\infty} \frac{Zt(n)}{(n-r)!} \qquad Z_8 = \sum_{n=r}^{\infty} \frac{Z(n)}{(n-r)!}$$

$$Zt_9 = \sum_{n} \frac{1}{\sum_{i=1}^{n} \frac{Zt(i)}{i!}} \qquad Z_9 = \sum_{n} \frac{1}{\sum_{i=1}^{n} \frac{Z(i)}{i!}}$$

$$Zt_{10}(a) = \sum_{n} \frac{1}{(Zt(n))^a \cdot \sqrt{Zt(n)!}} \qquad Z_{10}(a) = \sum_{n} \frac{1}{(Z(n))^a \cdot \sqrt{Z(n)!}}$$

$$Zt_{11}(a) = \sum_{n} \frac{1}{(Zt(n))^a \cdot \sqrt{(Zt(n)+1)!}} \qquad Z_{11}(a) = \sum_{n} \frac{1}{(Z(n))^a \cdot \sqrt{(Z(n)+1)!}}$$

Are these sums convergent to a constant value as the Smarandache function does [3]? If yes evaluate them. Are these constants (if they exist) irrational or trascendental?

Problem 36. *Evaluate the continued fraction [5] and radical [6] for the Pseudo-Smarandache and Pseudo-Smarandache-Totient numbers.*

Problem 37. *Is the number 0.1243549107585.... where the sequence of digits is Zt(n) for* $n \geq 1$ *an irrational or trascendental number? (We call this number the Pseudo-Smarandache-Totient constant).*

Problem 38. *Is the Smarandache Euler-Mascheroni sum (see chapter II for definition) convergent for Zt(n) and Z(n) numbers? If yes evaluate the convergence value.*

Problem 39. *Evaluate*

$$\sum_{k=1}^{\infty}(-1)^k \cdot Zt(k)^{-1} \quad \text{and} \quad \sum_{k=1}^{\infty}(-1)^k \cdot Z(k)^{-1}$$

Problem 40. *Evaluate*

$$\prod_{n=1}^{\infty}\frac{1}{Zt(n)} \quad \text{and} \quad \prod_{n=1}^{\infty}\frac{1}{Z(n)}$$

Problem 41 *Evaluate*

$$\lim_{k\to\infty}\frac{Zt(k)}{q(k)} \quad \text{and} \quad \lim_{k\to\infty}\frac{Z(k)}{q'(k)}$$

$$\text{where} \quad q(k)=\sum_{n\le k}\ln(Zt(n)) \quad \text{and} \quad q'(k)=\sum_{n\le k}\ln(Z(n))$$

Problem 42. *Evaluate*

$$F = p^{4\cdot S} \quad \text{where} \quad S=\sum_{k}\frac{1}{a(k)}$$

for the Smarandache (a(k)=S(k)), Pseudo Smarandache (a(k)=Z(k)) and Pseudo Smarandache-Totient (a(k)=Z(k)) numbers. Are these numbers F almost integers?

Problem 43. A*re there m, n, k non-null positive integers for which*

$$Zt(m\cdot n) = m^k \cdot Zt(n)\,?$$

Of course for m=1, the equation has infinite solutions because $Zt(1\cdot n)=Zt(n)$.

For n=1 we have $Zt(m\cdot 1)=m^k$ and then m will be a solution for k=1 if it is a fixed point of function Zt(n), that is if and only if Zt(m)=m.
The solutions (if they exist) for m>1 and n>1 are left to the reader.

Problem 44. *Let's indicate with FZt(n)=m the number of different integers k such that Zt(k)=n.*

Study the function FZt(n) and evaluate :

$$\lim_{m\to\infty}\frac{\sum_{k=1}^{m}\frac{FZt(k)}{k}}{m}$$

Problem 45. *Are there integers k>1 and n>1 such that* $Zt(n)^k = k\cdot Zt(n\cdot k)$*?*

Problem 46. *Study the convergence of the Pseuto-Smarandache-Totient harmonic series:*

$$\sum_{n=1}^{\infty}\frac{1}{Zt^{a}(n)} \qquad where\ \ "a"\ is\ \ a\ \ real\ \ number > 0$$

Problem 47. *Study the convergence of the series:*

$$\sum_{n=1}^{\infty}\frac{x_{n+1}-x_n}{Zt(x_n)}$$

where x_n *is any increasing sequence such that* $\lim\limits_{n\to\infty} x_n = \infty$

Problem 48. *Evaluate:*

$$\lim_{n\to\infty}\frac{\sum_{k=2}^{n}\frac{\ln(Zt(k))}{\ln(k)}}{n}$$

Is this limit convergent to some known mathematical constant?

Problem 49. *Solve the functional equation:*

$$Zt(n)^r + Zt(n)^{r-1} + \dots + Zt(n) = n \qquad where\ r\ is\ an\ integer \geq 2.$$

Let's indicate with N(x,r) the number of solutions of that equation for $n \le x$ *and fixed r. By a computer search the following result has been obtained:*

$$N(10^4,2) = 0$$

$$N(10^4,3) = 0$$

$$N(10^4,4) = 0$$

What about N(x,r) for $x>10^4$ *and r=2,3,4 ? What about N(x,r) for r>4?*
What about the functional equation:

$$Zt(n)^r + Zt(n)^{r-1} + \cdots + Zt(n) = k \cdot n \qquad \textit{where } r \textit{ and } k \in N \textit{ and } \ge 2 ?$$

Problem 50. *Is there any relationship between* $Zt(\prod_{k=1}^{m} m_k)$ *and* $\sum_{k=1}^{m} Zt(m_k)$?

Problem 51. *Solve the equation:*

$$\left\lfloor e^{\varphi(n)} \right\rfloor - Zt(n) = 0$$

For $1 \le n \le 5000$ *only one solution has been found: n=2. Is this solution unique?*

As already done for the Pseudo-Smarandache-Totient function other possible Pseudo-Smarandache functions can be defined.
In particular three of them will be introduced: the Pseudo-Smarandache-Squarefree, Pseudo-Smarandache-Prime and Pseudo Smarandache Divisor functions.
For the first one, theorems, conjectures and open questions are given too.

1.2 PSEDO-SMARANDACHE-SQUAREFREE FUNCTION

The Pseudo-Smarandache-Squarefree function Zw(n) is defined as the smallest integer m such that:

$$m^n$$

is divisbile by n, that is the value of m such that $\frac{m^n}{n}$ is an integer.

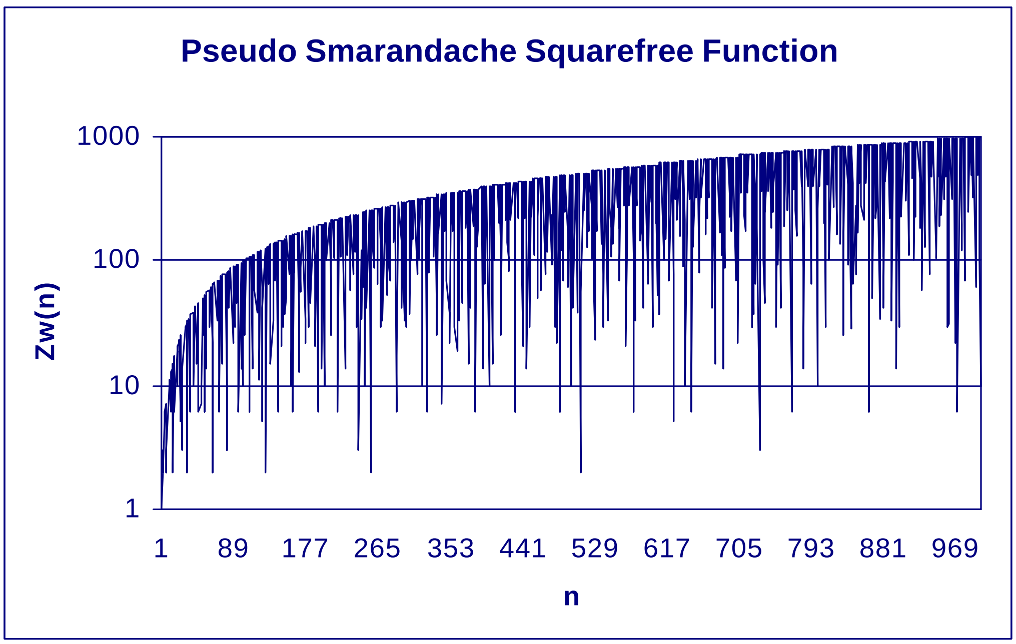

Fig. 1.3

A table of values of Zw(n) function for $1 \le n \le 100$ follows:

n	Zw(n)	n	Zw(n)	n	Zw(n)	n	Zw(n)	n	Zw(n)
1	1	21	21	41	41	61	61	81	3
2	2	22	22	42	42	62	62	82	82
3	3	23	23	43	43	63	21	83	83
4	2	24	6	44	22	64	2	84	42
5	5	25	5	45	15	65	65	85	85
6	6	26	26	46	46	66	66	86	86
7	7	27	3	47	47	67	67	87	87
8	2	28	14	48	6	68	34	88	22
9	3	29	29	49	7	69	69	89	89
10	10	30	30	50	10	70	70	90	30
11	11	31	31	51	51	71	71	91	91
12	6	32	2	52	26	72	6	92	46
13	13	33	33	53	53	73	73	93	93
14	14	34	34	54	6	74	74	94	94
15	15	35	35	55	55	75	15	95	95
16	2	36	6	56	14	76	38	96	6
17	17	37	37	57	57	77	77	97	97
18	6	38	38	58	58	78	78	98	14
19	19	39	39	59	59	79	79	99	33
20	10	40	10	60	30	80	10	100	10

An alternative definition of this function is given by F. Smarandache in [4]:

The largest square-free number dividing n (the square-free kernel of n).

Applying the notion of Smarandache continued fraction as reported in Castillo [5] and that of Smarandache continued radical as reported in Russo [6] to the Pseudo Smarandache squarefree numbers the following interesting convergence values are obtained. They have been calculated utilizing the Ubasic software package.

$$\frac{1}{3}+2\cdot\frac{\sqrt{3}}{\pi}\approx Zw(1)+\cfrac{1}{Zw(2)+\cfrac{1}{Zw(3)+\cfrac{1}{Zw(4)+\cfrac{1}{Zw(5)\ldots}}}}$$

$$\sqrt{Zw(1)+\sqrt{Zw(2)+\sqrt{Zw(3)+\sqrt{Zw(4)+\sqrt{Zw(5)+\ldots}}}}}\approx\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{5+\sqrt{6+\ldots}}}}}}$$

Let's now start to prove several theorems about the Zw(n) function.

Theorem 1.2.1 Zw(p)=p where p is any prime number.

Proof. In fact for n=p, m^p cannot be a multiple of p for $m \neq p$ because there isn't any common factor between m^p *and* p. On the contrary if m=p then:

$$p^p = k \cdot p \qquad where \quad k = p^{p-1}$$

Theorem 1.2.2 $Zw(p^a \cdot q^b \cdot s^c \ldots) = p \cdot q \cdot s \ldots$ where p, q, s... are distinct primes

Proof. Without loss of generality let suppose that $n = p^a \cdot q^b$.
Of course if m is a prime (equal to p or q) it is very easy to see that m^n cannot be a multiple of n. So the smallest value of m such that m^n is a multiple of n is given by the product $p \cdot q$.
In fact in this case:

$$m^n = k \cdot n \qquad where \quad k = p^{p^a \cdot q^b - a} \cdot q^{p^a \cdot q^b - b}.$$

Theorem 1.2.3 Zw(n)=n if and only if n is squarefree, that is if the prime decomposition of n contains no repeated factors. All primes of course are trivially squarefree [3].

Proof. The theorem is a direct consequence of theorem 1.2.1 and theorem 1.2.2 with $a = b = c \ldots = 1$.

Theorem 1.2.4 $Zw(n) \leq n$

Proof. Direct consequence of theorems 1.2.1, 1.2.2, and 1.2.3.

Theorem 1.2.5 $Zw(p^k) = p \quad for \;\; k \geq 1$ and p any prime.

Proof. Direct consequence of theorem 1.2.2 for q=s=....=1

According to the previous theorems, the Zw(n) function is very similar to the Mobius function [3].
In fact:

$$
Zw(n)=\begin{cases} n & \text{if } n \text{ is squarefree} \\ 1 & \text{if and only if } n=1 \\ \text{Product of distinct prime factors of } n & \text{if } n \text{ is not squarefree} \end{cases}
$$

Now we will prove the following two theorems related to the sum of Zw function and of its reciprocal respectively.

Theorem 1.2.6 $\sum_{k=1}^{\infty} \frac{Zw(k)}{k}$ diverges.

Proof. In fact:

$\sum_{k=1}^{\infty} \frac{Zw(k)}{k} > \sum_{p=2}^{\infty} \frac{Zw(p)}{p}$ where p is a prime number, and of course $\sum_{p=2}^{\infty} \frac{Zw(p)}{p}$ diverges because the number of primes is infinite [7] and $Zw(p) = p$.

Theorem 1.2.7 $\sum_{k=1}^{\infty} \frac{1}{Zw(k)}$ diverges

Proof. This theorem is a direct consequence of divergence of sum $\sum_{p} \frac{1}{p}$ where p is any prime number. In fact

$$
\sum_{k=1}^{\infty} \frac{1}{Zw(k)} > \sum_{p=2}^{\infty} \frac{1}{p}
$$

Theorem 1.2.8 The function Zw(n) is multiplicative, that is if GCD(m,n)=1 then $Zw(m \cdot n) = Zw(m) \cdot Zw(n)$.

Proof. Without loss of generality, let suppose that $m = p^{a} \cdot q^{b}$ and $n = s^{c} \cdot t^{d}$ where p, q, s, t are distinct prime numbers.
If GCD(m,n)=1 then:

$$
Zw(m \cdot n) = Zw(m) \cdot Zw(n)
$$

In fact $Zw(m\cdot n)=Zw(p^a\cdot q^b\cdot s^c\cdot t^d)=p\cdot q\cdot s\cdot t$ and $Zw(p^a\cdot q^b)=p\cdot q$, $Zw(s^c\cdot t^d)=s\cdot t$ according to the theorem 1.2.2.

On the opposite if $GCD(m,n)\neq 1$, let assume that $m=p^a\cdot q^b$ and $n=p^c\cdot s^d$. So according to the theorems 1.2.2 and 1.2.5 $Zw(m\cdot n)=Zw(p^a\cdot q^b\cdot p^c\cdot s^d)=p\cdot q\cdot s$ and $Zw(m)=Zw(p^a\cdot q^b)=p\cdot q$, $Zw(n)=Zw(p^c\cdot s^d)=p\cdot s$, that is $Zw(m\cdot n)\neq Zw(m)\cdot Zw(n)$

Theorem 1.2.9 The function Zw(n) is not additive, that is $Zw(m+n)\neq Zw(m)+Zw(n)$.

Proof. As an example we can consider the case $Zw(11+7)=Zw(18)=6\neq Zw(11)+Zw(7)=1$.
Anyway we can find numbers m and n such that the function Zw(n) is additive.
In fact if:

- m and n are squarefree
- k=m+n is squarefree

then Zt(n) is additive. In this case Zw(m+n)=Zw(k)=k and Zw(m)=m, Zw(n)=n according to theorems 1.2.1 and 1.2.3.

Theorem 1.2.10 $Zw(n)\geq 1 \quad for \quad n\geq 1$

Proof. This theorem is a direct consequence of definition of the Zw(n) function. In fact for n=1, the smallest m such that 1 divide Zw(1) is trivially 1. For $n\neq 1$, m must be greater than 1 because 1+n cannot be a multiple of n for any value of n.

Theorem 1.2.11 $0<\dfrac{Zw(n)}{n}\leq 1 \quad for \quad n\geq 1$

Proof. The theorem is a direct consequence of theorem 1.2.4 and 1.2.10.

Theorem 1.2.12 $\dfrac{Zw(n)}{n}$ is not distributed uniformly in the interval]0,1].

Proof. For n=1 and for any squarefree n, by definition the ratio is equal to 1.
If n is not a squarefree number, without loss of generality let suppose that $n=p^a\cdot q^b$ where p and q are two distict primes and $a\geq 1$, $b>1$ and viceversa.
There are two possibilities:

1. p=q In this case $\dfrac{Zw(n)}{n}=\dfrac{1}{p^{(a+b)-2}}\le\dfrac{1}{p}$ *and then* $\dfrac{Zw(n)}{n}\le\dfrac{1}{2}$

2. $p\neq q$ In this case $\dfrac{Zw(n)}{n}=\dfrac{p\cdot q}{p^{a}\cdot q^{b}}\le\dfrac{p\cdot q}{p^{2}\cdot q}=\dfrac{1}{p}$ *and then* $\dfrac{Zw(n)}{n}\le\dfrac{1}{2}$

So the ratio $\dfrac{Zw(n)}{n}$ is not distributed uniformly in the interval]0,1] because in the interval]1,1/2[doesn' t fall any point of the Zw(n) function.
Moreover from previous result it is obvious that for any value of $n\ge 1$ there is some m such that $\dfrac{Zw(n)}{n}=\dfrac{1}{m}$ for $m=1,2,3,4\ldots$
This result implies the following theorem where we prove that the ratio $\dfrac{Zw(n)}{n}$ can be made arbitrarily close to zero.

Theorem 1.2.13 For any arbitrary real number $e>0$, there is some number $n\ge 1$ such that $\dfrac{Zw(n)}{n}<e$.

Proof. Let’s form a product of distinct primes $q=p_1p_2\ldots\ p_k$ such that $\dfrac{1}{q}<e$ where e is any real number greater than zero. Now take a number n such that:

$$n=p_1^{a_1}\cdot p_2^{a_2}\ldots\ldots\ p_k^{a_k}\qquad \textit{where}\quad a_1,\ a_2,\ldots\ldots\ a_k\ge 2$$

By theorem 1.2.2,

$$\frac{Zw(n)}{n}=\frac{p_1\cdot p_2\ldots\ldots\ p_k}{p_1^{a_1}\cdot p_2^{a_2}\ldots\ldots\ p_k^{a_k}}=\frac{1}{p_1^{a_1-1}\cdot p_2^{a_2-1}\cdot\ldots\ldots\ p_k^{a_k-1}}\le\frac{1}{q}<e$$

Theorem 1.2.14 $Zw(p_k\#)=p_k\#$ where $p_k\#$ is the product of first k primes (primorial)[3].

Proof. The theorem is a direct consequence of theorem 1.2.3 being $p_k\#$ a squarefree number.

Theorem 1.2.15 The range of Zw function is the set of squarefree numbers.

Proof. A direct consequence of the fact that the function Zw applied to a squarefree number returns the squarefree itself and applied to a not squarefree number returns again a squarefree number (see theorems 1.2.1, 1.2.2, 1.2.3).

Theorem 1.2.16 The equation $\frac{Zw(n)}{n}=1$ has an infinite number of solutions.

Proof. The theorem is a direct consequence of theorem 1.2.1 and the well-known fact that there is an infinite number of prime numbers [7]

Theorem 1.2.17 We will use the notation FZw(n)=m to denote, as already done for the Zt(n) function, that m is the number of different integers k such that Zw(k)=n.

Example FZw(1)=1 since Zw(1)=1 and there are no other numbers n such that Zw(n)=1

Now we prove that for $1 \le n \le h$, where $h \in N$:

1. FZw(n)=0 for any n that is not a squarefree number
2. FZw(p)=a with $a=\left\lfloor \frac{\ln(h)}{\ln(p)} \right\rfloor$ where p is any prime number
3. FZw(1)=1
4. $$FZw(c)=1+\sum_{i=1}^{w(c)} \left\lfloor \frac{\ln(h)-\ln(c)}{\ln(p_i)} \right\rfloor + \sum_{s_1}\sum_{s_2}\ldots\sum_{s_{k-1}} W$$

where :

$$W=\left\lfloor \frac{\ln(h)-\ln(c)-s_1 \cdot \ln(p_2)-s_2 \cdot \ln(p_3)\ldots -s_{k-1} \cdot \ln(p_k)}{\ln(p_1)} \right\rfloor$$

- c is any composite squarefree number
- $w(c)$ is the number of distinct primes of c [3],
- and the sums with index s_i extend over all values of s_i for which $W \ge 1$

Proof. The items 1 and 3 are a direct consequence of definition of Zw(n) function. About the item 2, according to the theorem 1.2.5 the number of different integers k such that Zw(k)=p is given by the exponent *a* of the following inequality:

$$p^a \le h, \quad that\ is \qquad a \le \frac{\ln(h)}{\ln(p)}$$

Since a must be the largest integer $\le h$ then $a = \left\lfloor \frac{\ln(h)}{\ln(p)} \right\rfloor$.

Let's now prove the item 4. Without loss of generality let's suppose that $c = p_1 \cdot p_2$ where p_1 and p_2 are two distint primes.

According to the theorem 1.2.2 all the numbers n that have as distinct factors p_1 and p_2 will have the same Zw(n) value $(p_1 \cdot p_2)$.

So all the numbers of type $p_1 \cdot p_2 \cdot p_1^k$ such that their product is less or equal to h will have $(p_1 \cdot p_2)$ as Zw value. Then the number of different integers k such that Zw(k)=c is:

$$k = \left\lfloor \frac{\ln(h) - \ln(c)}{\ln(p_1)} \right\rfloor$$

A similar result holds for the product $p_1 \cdot p_2 \cdot p_2^k$. But we must consider also the product $p_1 \cdot p_2 \cdot p_1^{s_1} \cdot p_2^{s_2}$ for all values of s_1 and s_2 such that the product is less or equal to h.

So :

$$s_1 = \sum_{s_2} \left\lfloor \frac{\ln(h) - \ln(c) - s_2 \cdot \ln(p_2)}{\ln(p_1)} \right\rfloor$$

where the sum extend over all values of s_2 for which the ratio is ≥ 1. Moreover we need to add 1 because $Zw(c) = c = p_1 \cdot p_2$.

Theorem 1.2.18 The repeated iteration of the Zw(n) function will terminate always in a fixed point.

Proof. We can have three cases:

1. n=1
2. n= squarefree
3. n= not squarefree

For the case 1 of course Zw (1)=1. Same thing for the case 2 according to the theorem 1.2.3. For the case 3, according to the theorem 1.2.2, the application of function Zw to n will produce always a squarefree number and then the successive applications of Zw to it will produce always the same number being it squarefree.

> **Theorem 1.2.19** Both even and odd numbers are invariant under the application of the Zw function, that is if n is even (odd respectively) then Zw(n) is still even (odd respectively).

Proof. Let's suppose that $n = p_1^{a_1} \cdot p_2^{a_2} \ldots p_k^{a_k}$. Then $Zw(n) = p_1 \cdot p_2 \ldots p_k$. But if n is even (odd respectively) then the product of distinct prime factors also is even (odd respectively). So this prove the theorem.

> **Theorem 1.2.20** The diophantine equation Zw(n)=Zw(n+1) has no solutions.

Proof. In fact according to the previous theorem if n is even (odd respectively) then Zw(n) also is even (odd respectively). Therefore the equation Zw(n)=Zw(n+1) can not be satisfied because Zw(n) that is even should be equal to Zw(n+1) that instead is odd.

> **Theorem 1.2.21** The equations $\dfrac{Zw(n+1)}{Zw(n)} = k$ and $\left(\dfrac{Zw(n+1)}{Zw(n)}\right)^{-1} = k$ with k any positive integer and n>1 for the first equation don't admit solutions.

Proof. We must consider three cases:

1. n and n+1 are squarefree
2. n and n+1 are not squarefree
3. n is squarefree and n+1 is not squarefree and viceversa

For the case 1, without loss of generality let's assume that $n = p \cdot q$ and $n+1 = s \cdot t$ where p, q, s, t are distinct primes. Let's assume that $\dfrac{Zw(n)}{Zw(n+1)} = k$. Then $\dfrac{p \cdot q}{s \cdot t} = k$ according to the theorem 1.2.3.

But $s \cdot t - p \cdot q = 1$ being n and n+1 consecutive. This implies that $\dfrac{(s \cdot t - 1)}{s \cdot t} = k$ but this is absurd because $s \cdot t - 1$ and $s \cdot t$ don't have any common factor. So our initial assumption is false. Same thing for the equation $\dfrac{Zw(n+1)}{Zw(n)} = k$.

For case 2 without loss of generality let’s suppose that:

$$n = p^a \cdot q^b \quad and \quad n+1 = s^c \cdot t^d$$

where p, q, s, t are distinct primes.
According to the theorem 1.2.2, $Zw(n) = p \cdot q$ and $Zw(n+1) = s \cdot t$ and then let’s suppose that $\frac{s \cdot t}{p \cdot q} = k$.

Since $s \cdot t - p \cdot q = 1$, being n and n+1 consecutive, $m \cdot s \cdot t - m' \cdot p \cdot q = 1$ where m and m' are two integers, because $s^c \cdot t^d$ and $p^a \cdot q^b$ are multiple of $s \cdot t$ and $p \cdot q$ respectively. This implies that:

$$\frac{1 + m' \cdot p \cdot q}{m \cdot p \cdot q} = k$$

that is absurd because $1 + m' \cdot p \cdot q$ cannot be a multiple of $m \cdot p \cdot q$. So our initial assumption $\frac{Zw(n+1)}{Zw(n)} = k$ is not true. Same thing for the equation $\frac{Zw(n)}{Zw(n+1)} = k$
Analogously for the case 3.

Theorem 1.2.22 $\sum_{k=1}^{N} Zw(k) > \frac{6 \cdot N}{p^2}$ for any positive integer N.

Proof. The theorem is very easy to prove. In fact the sum of first N values of Zw function can be separated into two parts:

$$\sum_{m=1}^{N} Zw(m) + \sum_{l=4}^{N} Zw(l)$$

where the first sum extend over all m squarefree numbers and the second one over all l not squarefree numbers smaller or equal than N.
According to the Hardy and Wright result [3], the asymptotic number Q(n) of squarefree numbers $\leq N$ is given by:

$$Q(n) = \frac{6 \cdot N}{p^2}$$

and then:

$$\sum_{k=1}^{N} Zw(k) = \sum_{m=1}^{N} Zw(m) + \sum_{l=4}^{N} Zw(l) > \frac{6 \cdot N}{p^2}$$

because for theorem 1.2.3, Zw(m)=m and the sum of first N squarefree numbers is always greater or equal to the number Q(N) of squarefree numbers $\leq N$, namely:

$$\sum_{m=1}^{N} m \geq Q(N)$$

Theorem 1.2.23 $\displaystyle\sum_{k=1}^{N} Zw(k) > \frac{N^2}{2 \cdot \ln(N)}$ for any positive integer N.

Proof. In fact:

$$\sum_{k=1}^{N} Zw(k) = \sum_{m=1}^{N} Zw(m) + \sum_{p=2}^{N} Zw(p) > \sum_{p=2}^{N} Zw(p) = \sum_{p=2}^{N} p \text{ where p is any prime}$$

because by theorem 1.2.1, Zw(p)=p. But according to the result of Bach and Shallit [3], the sum of first N primes is asymptotically equal to:

$$\frac{N^2}{2 \cdot \ln(N)}$$

and this completes the proof.

Conjecture 1.2.1 The difference |Zw(n+1)-Zw(n)| is unbounded.

Let's suppose that $n = 2^{2^k}$ for $k \geq 0$. Then $n+1 = 2^{2^k} + 1$, namely (n+1) is a Fermat number [7].
If the Lehmer & Schinzel conjecture is true [7] then every Fermat number is squarefree.
This implies that $\left|Zw(n+1) - Zw(n)\right| = \left|2^{2^k} - 1\right| \approx 2^{2^k}$ according to the theorems 1.2.3 and 1.2.5 and then this difference can be as large as we want.

Of course if the Lehmer & Schinzel conjecture is true, then also the following conjecture will be true.

Conjecture 1.2.2 The Zw(n) function, is not a Lipschitz function, that is:

$$\frac{|Zw(m) - Zw(k)|}{|m-k|} \geq M \quad \text{where M is any integer [3].}$$

This conjecture is a direct consequence of the previous one where m=n+1 and k=n.

As for the Pseudo Smarandache Totient function let's now introduces some problems related to the Pseudo Smarandache Squarefree function.

Problem 1. *Which is the largest number k such that Zw(n), Zw(n+1), Zw(n+2)..... Zw(n+k) are all increasing (decreasing respectively) numbers?*

For the first 1000 values of Zw(n) the largest identified sequences have k=4 and k=3 respectively:

Example:

Zw(27) < Zw(28) < Zw(29) < Zw(30) < Zw(31)
Zw(422) > Zw(423) > Zw(424) > Zw(425)

Conjecture 1.2.3 The parameter k is upper limited.

Unsolved question. Find that upper limit.

Problem 2. *Solve the equation Zw(n)+Zw(n+1)=Zw(n+2)*

For the first 1000 values of Zw(n), six solutions have been found:
Zw(1)+Zw(2)=Zw(3), Zw(3)+Zw(4)=Zw(5), Zw(15)+Zw(16)=Zw(17),
Zw(31)+Zw(32)=Zw(33), Zw(127)+Zw(128)=Zw(129), Zw(255)+Zw(256)=Zw(257)

Is the number of solutions finite?

Problem 3. *Solve the equation Zw(n)=Zw(n+1)+Zw(n+2)*

For the first 1000 values of Zw(n) no solution has been found.
For the case n, n+1 and n+2 all squarefree numbers it is very easy to prove that the above equation cannot have solutions.

In fact without loss of generality let suppose that all three numbers are just the product of two distinct primes. Moreover let suppose that n is even. Then:

$$n = 2 \cdot p$$
$$n + 1 = s \cdot t$$
$$n + 2 = 2 \cdot q$$

where p, q, s, t are distinct primes. Of course being n and n+2 even the first prime factor must be 2.
Let's suppose now that the equation Zw(n)=Zw(n+1)+Zw(n+2) is satisfied. Then $2 \cdot p = s \cdot t + 2 \cdot q$.
Moreover the difference between n+2 and n+1 is 1 and therefore $s \cdot t = 2 \cdot q - 1$. This implies an absurd.
In fact $2 \cdot p = 4 \cdot q - 1$ cannot be true because $2 \cdot p$ is always even and $4 \cdot q - 1$ is always odd. So our initial assumption is false. Therefore if n, n+1, n+2 are all squarefree numbers the equation *Zw(n)=Zw(n+1)+Zw(n+2)* cannot have solutions.

Problem 4. *Find all the values of n such that* $Zw(n) = Zw(n+1) \cdot Zw(n+2)$

No solution has been found for the first 1000 values of Zw(n).
Is this true for all values of n?
Notice that if n is odd then the above equation cannot have solution. In fact according to the theorem 1.2.19, if n is odd then Zw(n) is odd and viceversa.
Since the product of $Zw(n+1) \cdot Zw(n+2)$ is always even if n is odd, it cannot be equal to n. Also if n, n+1 and n+2 are all squarefree then the equation has no solution.
In fact based on theorem 1.2.3 we should have $n = n^2 + 3n + 2$ that of course is an absurd.

Problem 5. *Solve the equation* $Zw(n) \cdot Zw(n+1) = Zw(n+2)$

Also in this case no solution has been found for the first 1000 values of Zw(n).
Is is this true for all values of n?
As for the problem 4, also in this case if n is odd the equation cannot have any solution. In fact if n, n+1 and n+2 are all squarefree numbers again the equation cannot have any solution because otherwise we should have $n^2 + n = n + 2$ that of course is absurd.

Problem 6. *Solve the equation* $Zw(n) \cdot Zw(n+1) = Zw(n+2) \cdot Zw(n+3)$

For the first 1000 values of Zw(n) no solution has been found. Is this true for all values of n ?

Problem 7. *Find all the values of n such that S(n)=Zw(n) where S(n) is the Smarandache function [1].*

The number of solutions is infinite. In fact the number of prime numbers is infinite and S(p)=p, Zw(p)=p. What happens for the composite numbers?

Problem 8. *Find the smallest k such that between Zw(n) and Zw(k+n), for n>1, there is at least a prime.*

Problem 9. *Find all the values of n such that Zw(Z(n))-Z(Zw(n))=0 where Z is the Pseudo Smarandache function [2].*

Problem 10. *Find all values of n such that Zw(Z(n))-Z(Zw(n))>0*

Problem 11. *Find all values of n such that Zw(Z(n))-Z(Zw(n))<0*

Problem 12. *Study the functions Zw(Z(n)), Z(Zw(n)) and Zw(Z(n))-Z(Zw(n)).*

Problem 13. *Is the number 0.12325672.... where the sequence of digits is Zw(n) for* $n \geq 1$ *an irrational or trascendental number? (We call this number the Pseudo-Smarandache-Squarefree constant).*

Problem 14. *Is the Smarandache Euler-Mascheroni sum (see chapter II for definition) convergent for Zw(n) numbers? If yes evaluate the convergence value.*

Problem 15. *Evaluate* $\sum_{k=1}^{\infty} (-1)^k \cdot Zw(k)^{-1}$

Problem 16. *Evaluate* $\prod_{n=1}^{\infty} \frac{1}{Zw(n)}$

Problem 17. *Evaluate* $\lim_{k \to \infty} \frac{Zw(k)}{q(k)}$ *where* $q(k) = \sum_{n \leq k} \ln(Zw(n))$

Problem 18. *Are there m, n, k non-null positive integers for which* $Zw(m \cdot n) = m^k \cdot Zw(n)$?

Problem 19. *Are there integers k>1 and n>1 such that* $Zw(n)^k = k \cdot Zw(n \cdot k)$ *?*

Problem 20. *Solve the problems from 28 up to 35 already formulated for the Zt function also for the Zw function.*

Problem 21. *Study the convergence of the Pseudo-Smarandache-Squarefree harmonic series:*

$$\sum_{n=1}^{\infty} \frac{1}{Zw^a(n)} \qquad \text{where } a > 0 \text{ and } a \in R.$$

Problem 22. *Study the convergence of the series:*

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{Zw(x_n)}$$

where x_n *is any increasing sequence such that* $\lim_{n \to \infty} x_n = \infty$

Problem 23. *Evaluate*

$$\lim_{n \to \infty} \frac{\sum_{k=2}^{n} \frac{\ln(Zw(k))}{\ln(k)}}{n}$$

Is this limit convergent to some known mathematical constant?

Problem 24. *Solve the functional equation:*

$$Zw(n)^r + Zw(n)^{r-1} + \ldots + Zw(n) = n \qquad \text{where } r \text{ is an integer } \geq 2$$

Let's indicate with N(x,r) the number of solutions of this equation for $n \leq x$ *and a fixed r. By a computer search we have found:*

N(1000,2)=0 N(1000,3)=0 N(1000,4)=0 N(1000,5)=0

Is this result true for all values of n and r? Wath about the functional equation:

$$Zw(n)^r + Zw(n)^{r-1} + \ldots\ldots + Zw(n) = k \cdot n \qquad \textit{where } k \textit{ and } r \textit{ are two integers } \geq 2$$

Problem 25. *Is there any relationship between* :

$$Zw\left[\prod_{k=1}^{m} m_k\right] \quad \textit{and} \quad \sum_{k=1}^{m} Zw(m_k) \ ?$$

Problem 26. *Study the Pseudo-Smarandache-Prime function Zp(n) defined as the smallest integer m such that:*

$$\sum_{k=1}^{m} p(k)$$

is divisible by n. Here p(k) is the k-th prime number [7]. By definition we force Zp(1)=1.

Problem 27. *Like the Pseudo-Smarandache-Prime function study the Pseudo-Smarandache-Divisor function Zd(n) defined as the smallest integer m such that:*

$$\sum_{k=1}^{m} d(k)$$

is divisible by n. Here d(k) is the number of divisors of k [3]. By definition Zd(1)=1.

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Chapter II

A set of new Smarandache-type notions in Number Theory.

In [1] F. Smarandache and later in [2] C. Dumitrescu and V. Seleacu defined a lot of sequences in number theory.
Several concepts of number theory can be applied to the set of Smarandache sequences as has already been done for example by Castillo [3] for Continued fractions and Russo [4] for Continued radicals.
In this chapter several new Smarandache-type-notions are introduced. Many open questions, examples and conjectures are given too.
Anyway there is a lot of room to investigate these subjects deeply and their connections with Number Theory.
We hope that this chapter will be a starting point for future investigations and developments.

1) Smarandache Zeta function

This function is an application of well known zeta function [5] of Number Theory to the set of Smarandache sequences a(n).

$$Sz(s)=\sum_{n=1}^{\infty}\frac{1}{a(n)^{s}} \quad where \quad s\in N$$

Please refer to the fig. 2.1 where the sum has been evaluated for the first 1000 terms of the power sequence. The Smarandache power sequence SP(n) is defined as the smallest m such that m^{m} is divisible by n [2].

Conjecture 2.1. The Smarandache Zeta function for the Smarandache power se

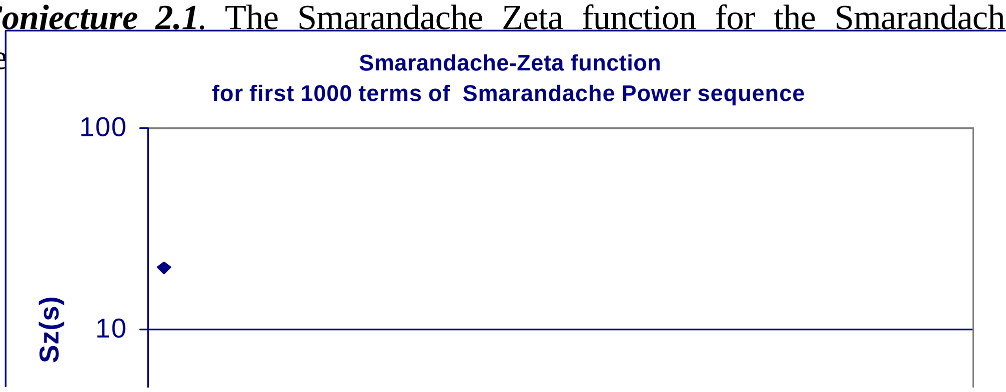

Fig. 2.1

Problem 1. *Is there any Smarandache sequence a(n) such that :*

$$\boldsymbol{p} \approx \sum_{k=1}^{\infty} \frac{b^{a(k)}-1}{c^{a(k)}} \cdot Sz(k+1) \qquad \textit{where} \quad b,c \in N$$

2) Smarandache sequence density

Let a(n) be a Smarandache sequence strictly increasing and composed of nonnegative integers. We can define the Smarandache sequence density as:

$$S_d = \lim_{n \to \infty} \frac{A(n)}{n}$$

where A(n) is the number of terms (in the Smarandache sequence) not exceeding n.

3) Smarandache Continued Radical

For any Smarandache sequence a(n) we can define the Smarandache continued radical

$$\sqrt{a(1)+\sqrt{a(2)+\sqrt{a(3)+\sqrt{a(4)+\cdots}}}}$$

As example see [4].

4) Smarandache generating function

Analogously to the well known definition in number theory of the generating function [5], for any Smarandache sequence a(n) we can define the following function Sf(x):

$$Sf(x) = \sum_n a(n) \cdot x^n$$

Problem 2. *Determine the generating function Sf for some of the most popular Smarandache sequences.*

8) Smarandache Euler-Mascheroni Sum

For any Smarandache sequence a(n) we can define the following sum used in number theory to define the Euler-Mascheroni constant [5]:

$$g = \lim_{m \to \infty} \sum_{k=1}^{m} \frac{1}{a(k)} - \ln(a(m))$$

Problem 3. *Is there any Smarandache sequence such that g is a constant?*

Conjecture 2.2. For the Smarandache power sequence [2] the following conjecture can be formulated:

$$g > \frac{1}{9} \cdot m^{\frac{3}{4}} \qquad \text{for all m values}$$

Problem 4. *For any Smarandache sequence that admits g as a constant, calculate:*

$$\lim_{n \to \infty} \frac{1}{a(n)} \cdot \prod_{i=1}^{n} \frac{1}{1 - \frac{1}{a(i)}}$$

Is this limit related to g?

Problem 5. *Is the following inequality satisfied*

$$\frac{1}{2\cdot a(n+1)} < \sum_{k=1}^{n}\frac{1}{a(k)} - \ln(a(n)) - g < \frac{1}{2\cdot a(n)}$$

for any Smarandache sequence a(n) that has g as constant?

9) Smarandache increasing/decreasing sequence

For any Smarandache sequence a(n) if $a(n+1)-a(n)>0$ for $n \ge n_0$, then a(n) is increasing for $n \ge n_0$.
Conversely, if $a(n+1)-a(n)<0$ for $n \ge n_0$ then a(n) is decreasing for $n \ge n_0$.

10) Smarandache A-sequence

An infinite Smarandache sequence a(n) of positive integers $1 \le a(1) \le a(2) \le a(3)\ldots$ is called an A-sequence if a(k) cannot be expressed as the sum of two or more distinct earlier terms of the sequence.

Example: The sequence given by concatenation of n copies of the Integer n: 1, 22, 333, 4444, 55555, ... [6] (ID=A000461) is a Smarandache A-sequence because $\sum_{k=1}^{n-1} a(k) < a(n)$ where a(k) is the k-th term of the sequence.

Problem 6. *Find further examples of Smarandache A-sequence.*

Problem 7. *Evaluate:*

$$S(A) = \sup_{all\,Smaran.\,A\;\;seq.} \sum_{k=1}^{\infty}\frac{1}{a(k)}$$

that is the largest reciprocal sum for the set of Smarandache A sequences.

11) Smarandache B2-sequence

An infinite Smarandache sequence b(n) of positive integers $1 \le b(1) \le b(2) \le b(3)\ldots$ is called a B2-sequence if all pairwise sums $b(i) + b(j),\ i \le j$ are distinct.

Problem 8. *Find some example of Smarandache B2-sequence*.

Problem 9. *Evaluate:*

$$S(B2) = \sup_{all\,Smaran.\ B2\ \ seq.} \sum_{k=1}^{\infty} \frac{1}{b(k)}$$

that is, the largest reciprocal sum for all Smarandache B2 sequences.

12) Smarandache nonaveraging sequence (or Smarandache C sequences)

An infinite Smarandache sequence c(n) of positive integers $1 \le c(1) \le c(2) \le c(3)\ldots$ is said nonaveraging if it contains no three terms in arithmetic progression.
That is, $c(i) + c(j) \ne 2 \cdot c(k)$ for any three distinct terms c(i), c(j) and c(k) forming the sequence.

Problem 10. *Find some examples of Smarandache nonaveraging sequence.*

Problem 11. *Evaluate:*

$$S(C) = \sup_{all\,Smaran.\ C\ \ seq.} \sum_{k=1}^{\infty} \frac{1}{c(k)}$$

that is, the largest reciprocal sum for all Smarandache C sequences.

Problem 12. *Is S(C) finite?*

13) Smarandache primitive sequence

A Smarandache sequence in which a term cannot be dived by any other ones.

Example: Smarandache prime-digital sequence, that is prime-digit numbers which are themselves prime: 2, 3, 5, 7, 23, 37, 53, ... [6] (ID=A019546)

14) Smarandache alternating series

A series of the form

$$\sum_k (-1)^k \cdot a(k)^{-1}$$

where a(k) is any Smarandache sequence. As example the alternate series has been calculated for the Smarandache square-product sequence [4].
It converges to -3.2536....

15) Smarandache Chebyshev Function

For any Smarandache sequence a(n) the following sum can be calculate analogously to the Chebyshev function defined in Number Theory [5]:

$$q(k) = \sum_{n \le k} \ln(a(n))$$

Problem 13. *Study for any Smarandache sequence a(k) the behaviour of the limit:*

$$\lim_{k \to \infty} \frac{a(k)}{q(k)}$$

For instance this limit has been evaluated for the Smarandache power sequence [2].
According to experimental data the following conjecture can be formulated:

$$\frac{a(k)}{q(k)} < \frac{4}{\sqrt{k}} \quad for\ \ k > 2$$

16) Smarandache Gaussian sum

For any Smarandache sequence a(n) we can define the following function Sg(m) analogously to the Gauss sum in number theory [5]:

$$Sg(m) = \sum_{n=0}^{m-1} e^{\frac{2 \cdot p \cdot i \cdot a(n)^2}{a(m)}}$$

The following graphs (fig. 2.2 and 2.3) show int(|Sg(m)|) versus m and its average value for the Smarandache product sequence [2].

17) Smarandache Dirichlet beta function

Analogously to the Dirichlet beta function [5], for any Smarandache sequence a(n) the following sum can be defined :

$$Sb(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2 \cdot a(n)+1)^s} \qquad where \quad s \in N$$

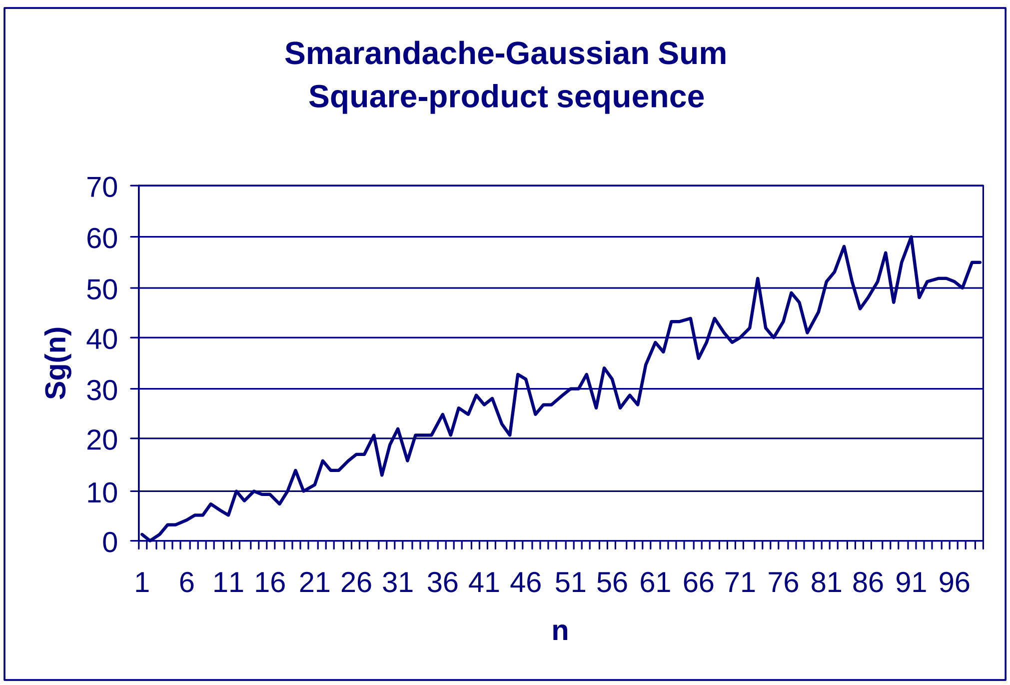

Fig. 2.2

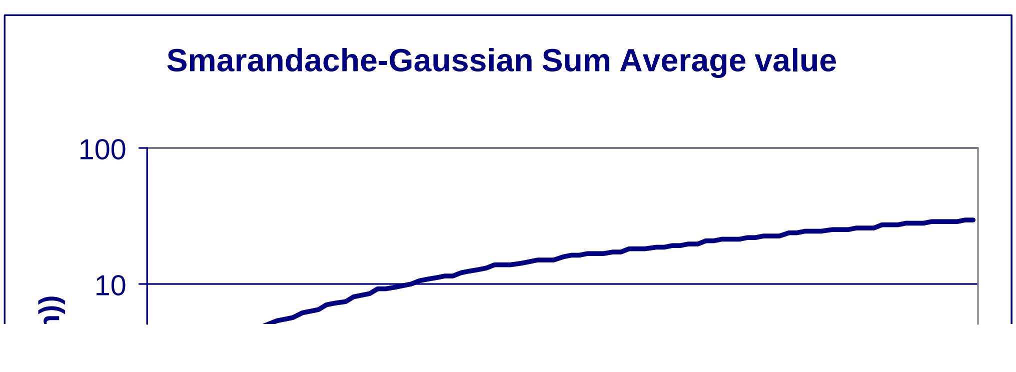

Fig. 2.3

The first ten Sb(s) values for the Smarandache power sequence [2] calculate using the first n=1000 terms of sequence follow:

s	Sb(s)
1	2.575537289795256
2	9.422351698410252e-2
3	-1.482843701206225e-2
4	-9.05618321791901e-3
5	-3.525313305129868e-3
6	-1.256320600490808e-3
7	-4.339450592280305e-4
8	-1.476728113473226e-4
9	-4.98400749475781e-5
10	-1.673920764594275e-5
11	-5.605398965109811e-6
12	-1.873680400189489e-6

Conjecture 2.3 Sb(s) for the Smarandache power sequence diverge for s<3.

Problem 14. *Is there any Smarandache sequence such that for a particular value of s, for instance* s_0, $Sb(s_0) = K$ *where K is the Catalan constant [5]?*

18) Smarandache Mobius Function

For any Smarandache sequence a(n) we can define the following function analogously to the Mobius function in number theory [5]:

$$Su(n)=\begin{cases} 0 & \text{if } a(n) \text{ has one or more repeated prime factors} \\ -1 & \text{if } a(n)=1 \\ (-1)^k & \text{if } a(n) \text{ is a product of } k \text{ distinct primes} \end{cases}$$

Problem 15. *For a given Smarandache sequence a(n) evaluate*

$$\lim_{n\to\infty} \sum_n \frac{Su(n)}{n^s} \qquad where \quad s \in N$$

Is there any Smarandache sequence a(n) such that its Smarandache Zeta function Sz(n) is related to this sum?

Problem 16. *Evaluate for some Smarandache sequence a(n):*

$$\sum_{n=1}^{\infty} \frac{Su(n)}{a(n)} \cdot \ln(a(n))$$

Is it convergent?

19) Smarandache Mertens Function

Analogously to the Mertens function in number theory [5] we can define the following function:

$$Sm(n) = \sum_{k=1}^{n} Su(k)$$

where Su(k) is the Smarandache Mobius function applied to any Smarandache sequence a(k).

As example see the fig. 2.4 where the function has been applied to Smarandache power sequence [2].

Problem 17. *Is there any Smarandache sequence such that the Mertens conjecture [5] is true?*

$$|Sm(n)| < \sqrt{a(n)}$$

Regar).

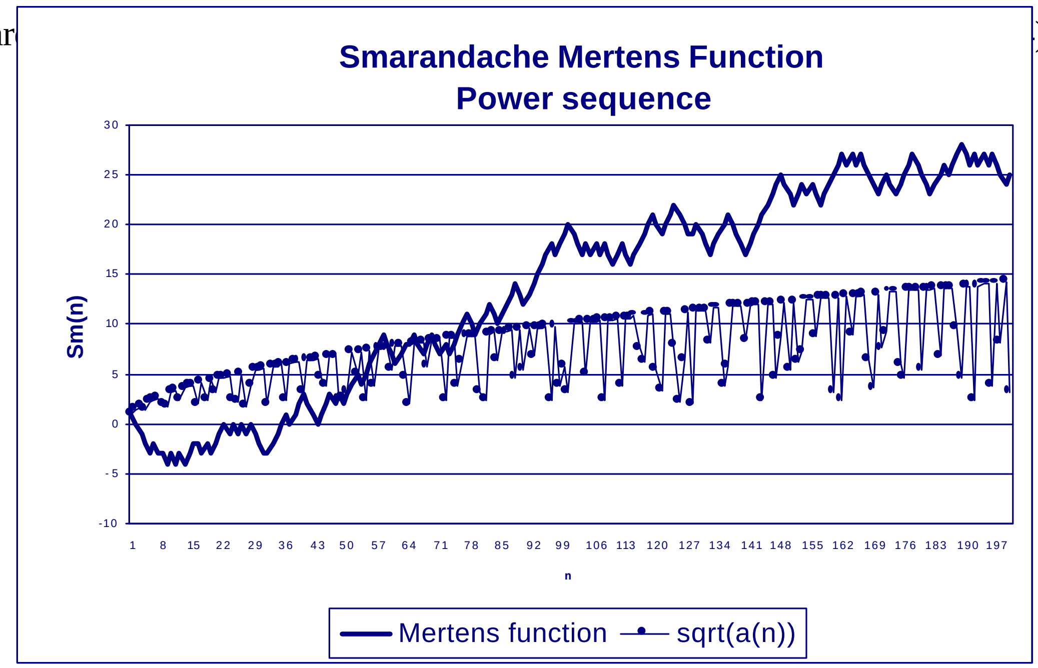

Fig. 2.4

20) Smarandache Ramanujan Constant

For any Smarandache sequence a(n) we can define the following function:

$$SR = e^{p \cdot \sqrt{\text{int}(p^{4 \cdot S})}}$$

where $S = \sum_n \frac{1}{a(n)}$ and assuming that S converge.

Examples (see [4] for sequences definition):

Sequence	S	SR
Square-product	0.728831....	260130672726.5336...
Cubic-product	0.615792....	30197683486.99318...
Factorial-product	0.913745....	7438763974956.754...
Palprime-product 1st	0.513624....	4442196022.5587.....
Palprime-product 2nd	1.239704....	883067941287422.3...

Problem 18. *Is there any Smarandache sequence a(k) like the cubic-product one such that SR is close to an integer?*

Notice also that the product:

$$F = \boldsymbol{p}^{4\cdot S}$$

can produce numbers almost integer.

Sequence	S	F
Square-product	0.728831....	70.994765.......
Cubic-product	0.615792....	59.983738.......
Factorial-product	0.913745....	89.007069.......
Palprime-product 1st	0.513624....	50.031646.......
Palprime-product 2nd	1.239704....	120.75843.......

If F is close to an integer we call it a Smarandache almost integer.

Problem 19. *Is there any Smarandache sequence a(n) that produce other Smarandache almost integers?*

21) Smarandache Dirichlet Eta Function

Analogously to the Dirichlet eta function in number theory [5], we can define the following function for any Smarandache sequence a(n):

$$Se(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{a(n)^s} \qquad where \quad s \in N$$

Conjecture 2.4 The Smarandache Dirichlet Eta function for the Smarandache power sequence [2] diverge for s<3.

22) Smarandache Dirichlet Lambda Function

Let a(n) be any Smarandache sequence. Analogously to the Dirichlet lambda function in number theory [5] the following function can be defined:

$$Sl(s) = \sum_{n=1}^{\infty} \frac{1}{(2 \cdot a(n) + 1)^s} \quad \text{where } s \in N$$

Conjecture 2.5 The Smarandache Dirichlet lambda function for the power sequence [2] diverge for s<5.

Problem 20. *Is there any Smarandache sequence a(n) such that:*

$$Sz(s) + Se(s) = 2 \cdot Sl(s)$$

for some $s \in N$*, where Sz(s) is the Smarandache Zeta function and Se(s) the Smarandache Dirichlet Eta function?*

Problem 21. *Is there any* $s \in N$ *such that for a particular Smarandache sequence Se(s), Sl(s) and Sb(s) are all related to* $\boldsymbol{p}$ *?*

23) Smarandache totient function

Using the well known Euler function in number theory [5] the following Smarandache totient function for any Smarandache a(n) sequence can be defined:

$$St(n) = \boldsymbol{j}(a(n))$$

that is the number of positive integers $\le a(n)$ which are relatively prime to a(n).

Conjecture 2.6 For the Smarandache power sequence [2] $\frac{S1}{S2}$ converge to zero for $n \to \infty$, where:

$$S1=\sum_{k=1}^{n}\left(\frac{1}{St(k)}\right)^{2} \qquad and \qquad S2=\left(\sum_{k=1}^{n}\frac{1}{St(k)}\right)^{2}$$

Problem 22. *For any Smarandache sequence a(n) evaluate*:

$$\frac{1}{N}\sum_{k=1}^{N}\frac{St(k)}{a(k)}$$

$$\lim_{n\to\infty}\sum_{n}\frac{1}{St(n)}$$

$$\lim_{n\to\infty}\sum_{n}St(n)$$

$$\frac{1}{N}\cdot\sum_{k=1}^{N}St(k)$$

Problem 23. *For a given Smarandache sequence a(n) evaluate*

$$\sum_{n=1}^{\infty}\frac{St(n)}{n^{s}} \quad where \quad s\in N$$

Is there any Smarandache sequence such that its Smarandache zeta function Sz(n) is related to this sum?

24) Smarandache divisor function

The Smarandache divisor function is defined as the number of the divisors d of a(n), where a(n) is any Smarandache sequence, that is:

$$Sd(n)=d(a(n))$$

As example see fig. 2.5 where the function Sd has been applied to the Smarandache power sequence [2].

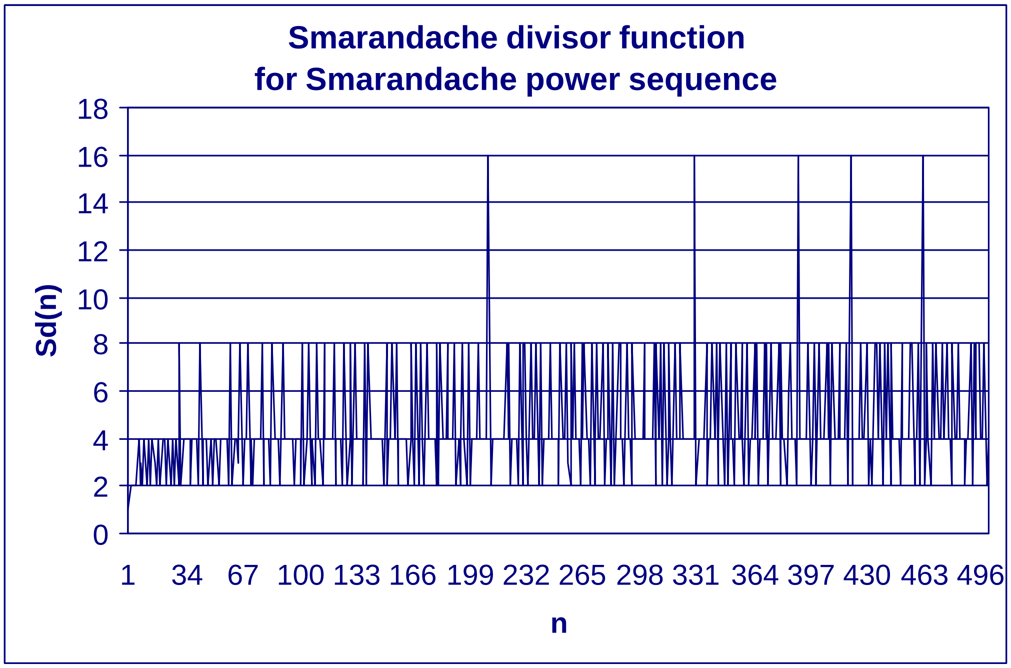

Fig. 2.5

Conjecture 2.7 For the Smarandache power sequence [2]:

$$\frac{\sum_{k=1}^{N} Sd(k)}{N} \approx g \cdot \ln(N) + 1$$

where g is the Euler-Mascheroni constant [5].

Problem 24. *Study* $\frac{E(Sd(n))}{n}$ *for any Smarandache sequence where E(Sd(n)) is the average value of Smarandache divisor function.*

Conjecture 2.8 For the Smarandache power sequence [2]:

$$\sum_{k=1}^{n} Sd(k) \approx \frac{7}{4} \cdot n^{a} \qquad\qquad where \;\; a \approx 1.154....$$

25) Smarandache Summatory Divisor function

For any Smarandache sequence a(n) the following function can be defined:

$$Ss(n) = \sigma(a(n))$$

where $\sigma(a(n))$ is the sum of the divisors of a(n) [5]. As example see the fig. 2.6. The function has been applied to the power sequence [2].

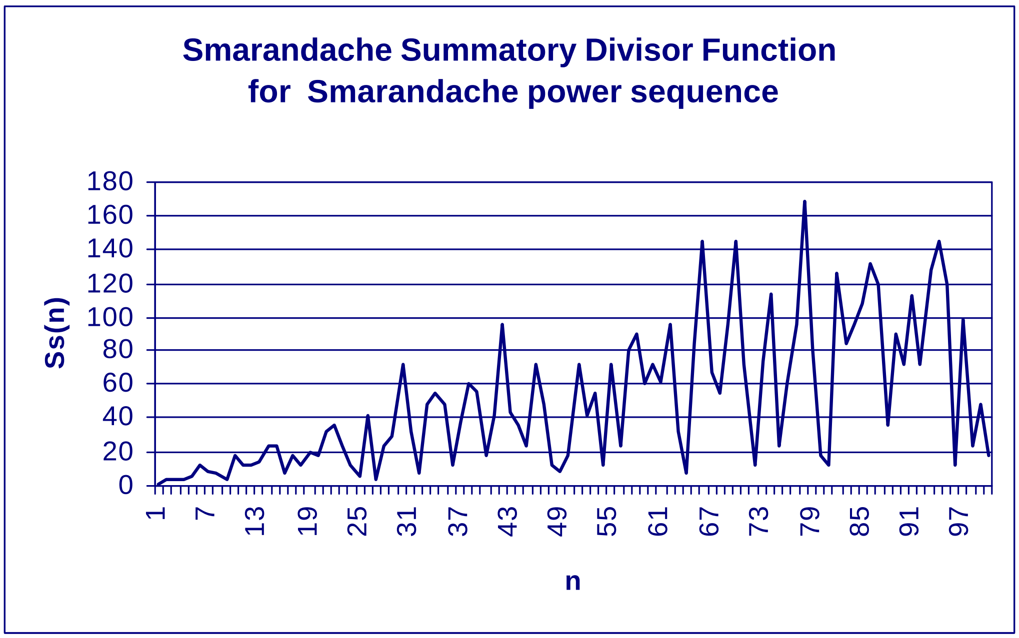

Fig. 2.6

Conjecture 2.9 For the Smarandache power sequence [2] the two following series are asymptotically equal to:

$$\sum_{k=1}^{\infty} Ss(k) \approx \frac{1}{2} \cdot n^{2}$$

$$\sum_{k=1}^{\infty} \frac{1}{Ss(k)} \approx a \cdot n^{\frac{1}{3}} \qquad where \quad a \approx 1.3275K$$

Problem 25. *Is there any Smarandache sequence a(n) such that the following inequality is satisfied?*

$$\frac{Ss(n)}{a(n) \cdot \ln(\ln(a(n)))} \leq e^{g} + \frac{2 \cdot (1-\sqrt{2}) + g - \ln(4 \cdot p)}{\sqrt{\ln(a(n))} \cdot \ln(\ln(a(n)))}$$

where g is the Smarandache Euler-Mascheroni constant for the sequence a(n). This question is well formulated if the hypothesis that the Smarandache sequence a(n) converges to a g constant value is satisfied.

Problem 26 . *For any Smarandache sequence evaluate:*

$$\frac{1}{N} \sum_{k=1}^{N} Ss(k)$$

$$\lim_{k \to \infty} \sum_{k} \frac{1}{Ss(k)}$$

$$\lim_{k \to \infty} \sum_{k} Ss(k)$$

26) Smarandache prime factors function

The Smarandache prime factors is defined for any Smarandache sequence a(n) as:

Spf(n)=r(a(n))

where r(a(n)) is the number of distinct prime factors of a(n) [5].

Conjecture 2.10 For Smarandache power-sequence [2]:

$$\frac{1}{N} \sum_{k=1}^{N} Spf(k) \approx \frac{24 \cdot \ln(2)}{p^{2}} \cdot N^{a} \qquad where \quad a \approx 0.0459....$$

Problem 27. *For any of Smarandache's sequences a(n) evaluate:*

$$\frac{1}{N}\sum_{k=1}^{N} Spf(k)$$

$$\lim_{k\to\infty}\sum_{k}\frac{1}{Spf(k)}$$

$$\lim_{k\to\infty}\sum_{k} Spf(k)$$

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Chapter III

A set of new Smarandache sequences.

The greatest discovery of all humankind may well have been the natural numbers. For many past aeons, philosophers and mathematicians have studied the sequence of natural numbers, uncovering startling and mystifying truths. Mathematics, itself, began with the natural numbers and the study of their relationships.
The demand for solutions to new and sophisticated problems forced us to use the natural numbers to construct the great edifice of modern mathematics.
Mathematicians, in their search for these solutions, progressed beyond the natural numbers, discovering rationale, irrationale, transcendetal and complex numbers.
However, the majority of the mathematicians recognize that the most important problems in mathematics today still involve the natural number sequences.
In this chapter several new Smarandache sequences are introduced.
In the spirit of F. Smarandache several open questions and problems are presented too.
Each sequence is labeled with an increasing initial number.

1) Smarandache repeated digit sequence with 1-endpoints

111, 1221, 13331, 144441, 1555551, 16666661, 177777771, 1888888881, 19999999991, 1101010101010101010101,.....

Starting with n repeat it n times and then add 1 on the left and on the rigth.

- How many terms are primes?
- Observe that the sum of digits of each term is given by n^2+2. Then the n-th term will be not a prime if $\frac{n^2+2}{3}$ is an integer.

2) Smarandache repeated digit generalized sequence with n-endpoints

n1n, n22n, n333n, n4444n, n55555n, n666666n, n7777777n, n88888888n, ...

where n is any positive integer.

- Determine the general expression for a(n) in terms of n.

3) Smarandache alternate consecutive and reverse sequence

1, 21, 123, 4321, 12345, 654321, 1234567, 87654321, 123456789, 10987654321, 1234567891011, ……

The odd terms are the consecutive integers starting with 1 on the left. The even ones are the consecutive integers but reversed (namely starting with 1 on the rigth).

- How many primes are there in this sequence?
- Note that in this sequence the n-th term has n digits. The trailing digits of the terms follow the sequence 1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 1, 1, 3, 1, 5, 1, 7......
- Of course by definition the sum of the digits of each term is given by $\frac{m(m+1)}{2}$

4) Smarandache alternate consecutive and reverse primes sequence (SACRP)

2, 32, 235, 7532, 235711, 13117532, 2357111317, 1917117532, 23571113171923

The odd terms are the consecutive primes starting with 2 on the left. The even ones are always the consecutive primes but reversed, namely starting with 2 on the right.

- How many terms are primes?
- Note that the sum of digits of some term is a prime. How many terms are prime and the sum of their digits is prime too?
- We define as "additive primes" a prime number whose digits sum is prime too (see sequence #11).
- Is there any perfect square among the terms of that sequence? A perfect cube?

5) Smarandache alternate consecutive and reverse palprimes sequence (SACRPP)

2, 32, 235, 7532, 235711, 101117532, 235711101121, 131121101117532, 235711101121131151, 181151131121101117532, ...

Same as sequence #4, but using the palindromic primes [1] instead of primes.

- How many terms are primes?
- Evaluate $\lim_{n\to\infty} \frac{a(n+1)}{a(n)}$ where a(n) is the n-th term of sequence.
- Evaluate the continued general fraction [2]:

$$a(1)+\cfrac{b(1)}{a(2)+\cfrac{b(2)}{a(3)+\cfrac{b(3)}{a(4)+\cfrac{b(4)}{a(5)+\dots}}}}$$

where a(n) is the SACRPP sequence and b(n) the SACRP sequence.

6) Smarandache alternate consecutive and reverse Fibonacci sequence

1, 11, 112, 3211, 11235, 853211, 11235813, 2113853211, 112358132134, ...

Again as for the sequences #4 and #5 but this time using the Fibonacci sequence.

- Evaluate $\lim_{n\to\infty} \frac{a(n)}{a(n+1)}$. Is this limit convergent?

- Is any terms of that sequence a Fibonacci number itself?
- Is this sequence a Smarandache A and C sequence (see chapter II for definition)?
- Let N+ be the number of terms even and N- the number of terms odd. Evaluate

$$\lim_{n\to\infty} \frac{N-}{N+}$$

7) Smarandache alternate consecutive and reverse generalized sequence

Let a(n) be a sequence with $n \geq 1$. The Smarandache alternate and reverse generalized sequence is given by:

$$a(1),\ \overline{a(2)a(1)},\ \overline{a(1)a(2)a(3)},\ \overline{a(4)a(3)a(2)a(1)} \ldots\ldots$$

namely by the alternate concatenation of terms of a(n).

8) Smarandache additive Fibonacci sequence

1, 1, 2, 3, 5, 8, 21, 233, 317811, 3524578, ...

Sequence of Fibonacci numbers whose digits sum is a Fibonacci number too.

- Let <N+> be the average value of all even terms of this sequence and <N-> that of odd terms.

  Evaluate $\lim_{n \to \infty} \left| < N+ > - < N- > \right|$.

  Is this difference convergent or divergent?
  If it converges, find the limit.

- Evaluate $\lim_{n \to \infty} \frac{< N- >}{< N+ >}$. Is this ratio convergent or divergent?

- If it converges, find the limit.
- Is that sequence finite?

9) Smarandache additive Square sequence

1, 4, 9, 36, 81, 100, 121, 144, 169, 196, 225, 324, 400, 441, 484, 529, 900, 961, …

Sequence of squares which digits sum is a square too.

- Is this sequence finite?
- Which is the maximum number of consecutive square that are also additive square? (In the first terms reported above this maximum number is 7).

10) Smarandache additive Cubic sequence

1, 8, 125, 512, 1000, 1331, 8000, 19683, 35937, ...

Sequence of cubes which digits sum is a cube too.

- Is this sequence finite?
- Which is the maximum number of consecutive cube that are also additive cube?
- Evaluate the continued general fraction :

$$a(1)+\cfrac{b(1)}{a(2)+\cfrac{b(2)}{a(3)+\cfrac{b(3)}{a(4)+\cfrac{b(4)}{a(5)+\ldots}}}}$$

where a(n) is the Smarandache additive square sequence and b(n) the Smarandache additive cubic sequence.

11) Smarandache additive generalized sequence

Let a(n) be a sequence with $n \geq 1$ and let be $a(n) = x_1 x_2 \ldots x_k$ in base 10.
Then a(n) is a term of Smarandache additive sequence if and only if:

$$x_1 + x_2 + \ldots\ \ldots + x_k = a(m) \quad where\ \ m \geq 1$$

namely if $x_1 + x_2 + \ldots\ \ldots + x_k$ belongs to the original sequence a(n).

- Study the Smarandache additive prime sequence, that is the sequence of primes which sum of digits is a prime.
- Is the Smarandache additive prime sequence finite?
- Find the largest sequence of consecutive additive primes.
  (Today the largest known sequence of this type contains 15 terms and start with the prime 2442113).

12) Smarandache multiplicative square sequence

1, 4, 9, 49, 144, 289, ...

Sequence of squares which product of digits is a square too.

- Is that sequence finite?
- How many terms in that sequence are also additive squares?

13) Smarandache multiplicative Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, ...

Sequence of Fibonacci numbers which digit's product is a Fibonacci number too.

- Which is the maximum number of consecutive Fibonacci numbers that are also Fibonacci-additive?
- Is that sequence finite?

Conjecure: Yes.

- How many terms are primes?

14) Smarandache multiplicative generalized sequence

Let a(n) be a sequence with $n \geq 1$ and let be $a(n) = x_1 x_2 \ldots x_k$ in base 10.
Then a(n) is a term of Smarandache multiplicative sequence if and only if:

$$x_1 \cdot x_2 \cdot \ldots \ldots \cdot x_k = a(m) \quad where \;\; m \geq 1$$

that is if $x_1 \cdot x_2 \cdot \ldots \ldots \cdot x_k$ belongs to the original sequence a(n).

- Study the Smarandache multiplicative cubic sequence
- Study the Smarandache multiplicative prime sequence.
- Is the Smarandache multiplicative prime sequence finite?
- How many terms in the Smarandache multiplicative prime sequence are palindromic primes?

15) Smarandache additive and multiplicative generalized sequence

Let a(n) be a sequence with $n \geq 1$ and let be $a(n) = x_1 x_2 \ldots x_k$ in base 10.
a(n) is a term of Smarandache additive and multiplicative sequence if and only if $x_1 \cdot x_2 \cdot \ldots \ldots \cdot x_k$ and $x_1 + x_2 + \ldots \ldots + x_k$ belong to the original sequence a(n).

- Study the Smarandache additive and multiplicative Fibonacci sequence.
- Is the Smarandache additive and multiplicative Fibonacci sequence finite?
- Study the Smarandache additive and multiplicative prime sequence.
- Is the Smarandache additive and multiplicative prime sequence finite?
- How many terms are palindromic primes?

16) Smarandache $\overline{nn^2}$ sequence

11, 24, 39, 416, 525, 636, 749, 864, 981, 10100, 11121, 12144, ...

The sequence is formed concatenating n and n^2 for n=1, 2, 3, 4.....

The n-th term of the sequence is given by $a(n) = n \cdot 10^{d(n^2)} + n^2$ where $d(n^2)$ is the number of digits of n^2.

- How many terms of that sequence are palindromes?
- How many primes are in that sequence?

Conjecture: A finite number.

- How many terms of that sequence are perfect square?

Conjecture: None.

- How many terms of that sequence are squarefree?

17) Smarandache $\overline{nn^3}$ sequence

11, 28, 327, 464, 5125, 6216, 7343, 8512, 9729, 101000, 111331, 121728, 132197, 142744, 153375, 164096 ......

The sequence is formed concatening n and n^3 for n=1, 2, 3, 4.....

The n-th term of the sequence is given by $a(n) = n \cdot 10^{d(n^3)} + n^3$ where $d(n^3)$ is the number of digits of n^3.

- How many terms in this sequence are palindromes?
- How many primes are in that sequence?

Conjecture: A finite number.

- How many terms in that sequence are perfect cube?
- How many terms in that sequence are squarefree?
- Evaluate $\lim_{n \to \infty} \frac{a(n+1)}{a(n)}$ where a(n) is the n-th term of the sequence.

18) Smarandache $\overline{nn^m}$ generalized sequence

The sequence is obtained concatenating n and n^m with $m \geq 1$.

19) Smarandache $\overline{n2n}$ sequence

12, 24, 36, 48, 510, 1122, 1326, 1428, 1530, 1632, 1734, 1836, 1938 ......

For each n, the n-th term of the sequence is formed concatenating n and $2 \cdot n$.

The n-th term of sequence is given by $a(n) = 2 \cdot n + n \cdot 10^{d(2n)}$ where d(2n) is the number of digits of 2n.

- How many terms in that sequence are perfect squares?

Conjecture: A finite number.

20) Smarandache $\overline{nkn}$ generalized sequence with k>1

For each n, the n-th term of sequence is formed concatenating together n and $k \cdot n$.

$$\overline{1k1}, \overline{2k2}, \overline{3k3}, \overline{4k4}, \overline{5k5}, \ldots\ldots$$

21) Smarandache multiple sequence

1, 24, 369, 481216, 510152025, 61218243036, 7142128354249, 816243240485664

The nth term is obtained concatenating together $n, 2\cdot n, 3\cdot n, \ldots\ldots n \cdot n$.

- Evaluate for that sequence the simple continued fraction and the continued radical.
- How many terms are primes?
- Evaluate $\lim_{k\to\infty} \sum_k \frac{a(k+1)}{a(k)}$ where a(n) is the n-th term of Smarandache multiple sequence.

22) Smarandache $\overline{nZt(n)}$ sequence

11, 22, 34, 43, 55, 64, 79, 810, 97, 105, 118, 126, 1346, 149, 1519, 1610, 1718, 187,
1960, 2016, 2111, 228 ......

The sequence is obtained concatenating n and Zt(n), where Zt(n) is the Pseudo Smarandache Totient function.
Note that the n-th term is given by: $a(n) = n \cdot 10^{d(Zt(n))} + Zt(n)$ where d(Zt(n)) is the number of digits of Zt(n)

- How many terms are primes?

Conjecture: An infinte number.

- Are there k (k>1) terms in this sequence in arithmetical progression? (for example in these first ones there are 2 terms: 11 and 22)
- Which is the largest k value?
- Is |a(n+1)-a(n)| bounded or unbounded?
- Is the Smarandache series $\sum_{n=1}^{\infty} \frac{1}{a(n)}$ convergent? If yes evaluate the limit.
- If we indicate with N(+) the number of terms even and with N(-) the number of terms odd, evaluate $\lim_{n \to \infty} |N(+) - N(-)|$
- How many terms are perfect square?

  Conjecture: A finite number.

- How many terms are perfect cube?

Conjecture: None

23) Smarandache $\overline{nf(n)}$ generalized sequence

Let f(n) be any number-theoretic function. Then the sequence is generated by concatenating n and f(n).

$\overline{1f(1)}, \overline{2f(2)}, \overline{3f(3)}, \overline{4f(4)}, \overline{5f(5)}, \overline{6f(6)}, \ldots\ldots$

- Study the Smarandache $\overline{nZw(n)}$ sequence where Zw(n) is the Pseudo Smarandache Squarefree function (see chapter I).
- Study the Smarandache $\overline{nZ(n)}$ sequence where Z(n) is the Pseudo Smarandache function [6].
- Study the Smarandache $\overline{nS(n)}$ sequence where S(n) is the Smarandache function [5].

24) Smarandache left-rigth natural numbers sequence

1, 21, 213, 4213, 42135, 642135, 6421357, 86421357, 864213579, 10864213579, 1086421357911, 121086421357911 ....

Starting with 1 add first on the left and then on the rigth the natural numbers.

- The n-th term has n digits and their sum is equal to n(n+1)/2.
- Observe that the trailing digits of the terms follow the sequence 1 1 3 3 5 5 7 7 9 9
- Determine the expression of a(n) in terms of n.
- How many terms are primes?
- Let's observe that the terms of the sequence a(n) for $n = (2 \cdot k + 1) \cdot 5$ and $n = (2 \cdot k + 1) \cdot 5 + 1$ (where $k \in N$) have the last digit equal to 5 and then can not be prime. Also the terms which sum of digits is a multiple of 3 can not be a prime, that is if for the n-th term of the sequence $\frac{n(n+1)}{2} = 3 \cdot m$ where m is a natural number.

25) Smarandache rigth-left natural numbers sequence

1, 12, 312, 3124, 53124, 531246, 7531246, 75312468, 975312468, 97531246810

Starting with 1 add first on the rigth and then on the left the natural numbers.

- The trailing digits of the terms follow the sequence 2 2 4 4 6 6 8 8 0 0 and only the first term is equal to 1.
- Evaluate the continued radical and continued fraction for this sequence.

26) Smarandache left-rigth prime sequence

2, 32, 325, 7325, 732511, 13732511, 1373251117, 191373251117,
19137325111723, 2919137325111723, ...

Starting with the first prime number 2 add first on the left and then on the rigth the successive primes.

- The digits sum of the n-th term is approximatively $\frac{n^2}{2 \cdot \ln(n)}$.
- How many terms are primes?
- How many terms are additive primes?

27) Smarandache rigth-left prime sequence

2, 23, 523, 5237, 115237, 11523713, 1711523713, 171152371319, 23171152371319, 2317115237131929, ...

Starting with the first prime number 2 add first on the rigth and then on the left the successive primes.

- How many terms are primes?
- Let a(n) the Smarandache left-rigth prime sequence and b(n) the Smarandache rigth-left prime sequence. Evaluate the Smarandache general continued fraction [2]:

$$a(1)+\cfrac{b(1)}{a(2)+\cfrac{b(2)}{a(3)+\cfrac{b(3)}{a(4)+\cfrac{b(4)}{a(5)+\dots}}}}$$

28) Smarandache rigth-left generalized sequence

Let a(n) be any sequence with $n \geq 1$. Then the Smarandache rigth-left sequence is given by following concatenations:

$a(1)$, $\overline{a(1)a(2)}$, $\overline{a(3)a(1)a(2)}$, $\overline{a(3)a(1)a(2)a(4)}$, $\overline{a(5)a(3)a(1)a(2)a(4)}$ ....

- How many terms belong to the original sequence a(n) besides a(1)?

29) Smarandache left-rigth generalized sequence

Let a(n) be any sequence with $n \geq 1$. Then the Smarandache left-rigth sequence is given by following concatenations:

$a(1)$, $\overline{a(2)a(1)}$, $\overline{a(2)a(1)a(3)}$, $\overline{a(4)a(2)a(1)a(3)}$, $\overline{a(4)a(2)a(1)a(3)a(5)}$ ....

- How many terms belong to the original sequence a(n) besides a(1)?

30) Smarandache $p^m + p^s$ generalized sequence

How many times a natural number n can be written as sum of $p^m + p^s$ where p is prime and $m, s \geq 1$

- Study the sequence generated with $m = s = 2$.
- Evaluate the continued radical [3] for the above sequence.
- Study the sequence generated with m=2 and s=3.
- Is the number 0.00000000000100001..... where the sequence of digits is the above sequence, irrational?
- Evaluate the continued radical [3] for that sequence.

31) Smarandache $n^m + n^s$ generalized sequence

How many times a natural number n can be written as a sum of $n^m + n^s$ where s and m are positive integers.

- Sudy the sequence with s=2 and m=3.
- Which is the maximum value of a(n) for the previous sequence?
  For n<=8000 via a computer program searching the maximum has been found at n=1025 with a(1025)=4
- Regarding that sequence evaluate the continued radical.

32) Smarandache embedded even generalized sequence

Let a(n) with $n \geq 1$ be a sequence and let's suppose that in base 10, $a(n) = x_1 x_2 \ldots x_k$.

A Smarandache embedded even sequence is the number of distinct even numbers formed with the digits $x_1 x_2 \ldots x_k$ of a(n).

- Study the Smarandache embedded even sequence e(n) for the sequence a(n) of the natural numbers. Which is the maximum value of e(n)?
- Evaluate the asymptotic average value for that sequence.

- Find an expression for e(n) in terms of n.
- Study the Smarandache ebedded even sequence e(n) for the sequence a(n) of the Fibonacci numbers. Which is the maximum values of e(n)?

33) Smarandache embedded odd generalized sequence

Let a(n) with $n \geq 1$ be a sequence and let's suppose that in base 10, $a(n) = x_1 x_2 \ldots x_k$.
A Smarandache embedded odd sequence is the number of distinct odd numbers formed with the digits $x_1 x_2 \ldots x_k$ of a(n).

- Study the Smarandache odd embedded sequence o(n) for the sequence a(n) of the natural numbers.
- Find an expression for o(n) in terms of n.
- Which is the maximum value of o(n)?
- Evaluate the asymptotic average value of that sequence.
- Study the Smarandache embedded odd sequence for the sequence of the Fibonacci numbers.

34) Smarandache embedded primes generalized sequence

Let a(n) with $n \geq 1$ be a sequence and let suppose that in base 10, $a(n) = x_1 x_2 \ldots x_k$.

A Smarandache embedded primes sequence is the number of distinct primes numbers formed with the digits $x_1 x_2 \ldots x_k$ of a(n).

- Study the Smarandache embedded primes sequence p(n) for the sequence of natural numbers.
- Which is the maximum value of p(n)?
- Evaluate the continued radical for this sequence.
- Study the Smarandache embedded primes sequence for the sequence a(n) of the Fibonacci numbers.
- Study the Smarandache embedded primes sequence for the sequence a(n) of the prime numbers.

35) Smarandache generalized powers primes sequence

Let a(n) with $n \geq 1$ be a sequence. The Smarandache powers primes sequence is the sequence of primes of the form $a(n)^m + a(t)^s$ with m, n, s and t positive integers.

- Study the Smarandache Fibonacci powers primes sequence a(n) for m=2 and s=2, that is the sequence of prime numbers of the form $F(j)^2 + F(i)^2$ where F(i) is the i-th term of the Fibonacci sequence.
- How many terms are Fibonacci numbers?
- Evaluate $\lim_{n\to\infty} \frac{a(n)}{a(n+1)}$
- Is that sequence finite?
- Study the Smarandache Fibonacci power primes sequence for m=2 and s=3.
- Is that sequence finite?
- How many terms are Fibonacci numbers?

36) Smarandache pseudo Q numbers

0, 1, 1, 1, 3, 7, 9, 4, 5, 10, 16, 17, 11, 5, 12, 20, 23, 17, 9, 20, 18, 13, 14, 33, 43, 31, 28, 27, 2, 29, 4, 57, 34, 73, 33, 2, 34, 6, 71, 114, 90, 4, 71, 116, 117, 28, 9, 91, 248, 161, 4

Numbers generated by $a(n) = a(|n - a(n-1)|) + a(|n - a(n-2)|) + a(|n - a(n-3)|)$ for n>3 and where a(0)=0, a(1)=a(2)=a(3)=1

- Show these numbers a chaotic behavior like the Q-numbers (for definition of Q numbers see [1]) ?
- Within the chaotic behaviour is there some signs of order ? (namely this numbers exhibit approximate Period Doubling, Self-Similarity and Scaling as Q-numbers ?)

37) Smarandache-Cullen generalized primes sequence

Let f(n) be any number-theoretic function with $n \geq 1$. The Smarandache-Cullen primes sequence is the sequence of primes of the form $n \cdot 2^{f(n)} + 1$.

- Study the Smarandache-Cullen totient primes sequence, that is the sequence of primes of the form $n \cdot 2^{j(n)} + 1$, where $j(n)$ is the totient function [1].
- Is that sequence finite?

Conjecture: The sequence is infinite.

- Study the Smarandache-Cullen primes sequence, that is the primes of the form $n \cdot 2^{S(n)} + 1$ where S(n) is the Smarandache function [5].
- Study the Pseudo-Smarandache-Cullen primes sequence, that is the primes of the form $n \cdot 2^{Z(n)} + 1$ where Z(n) is the Pseudo Smarandache function [6].
- Study the Smarandache-Cullen-Squarefree primes sequence, that is the primes of the form $n \cdot 2^{Zw(n)} + 1$ where Zw(n) is the Pseudo Smarandache Squarefree function.

38) Smarandache-Pseudo-Cullen generalized primes sequence

Let f(n) be any number-theoretic function with $n \geq 1$. The Smarandache-Cullen primes sequence is the sequence of primes of the form $n \cdot 2^{f(n)} - 1$.

- Study the Smarandache-Pseudo-Cullen totient primes sequence, that is the sequence of primes of the form $n \cdot 2^{j(n)} - 1$ where $j(n)$ is the totient function [1].
- Is that sequence finite?
- How many palindromic primes are in that sequence?
- Study the Smarandache-Pseudo-Cullen primes sequence, that is the primes of the form $n \cdot 2^{S(n)} - 1$ where S(n) is the Smarandache function [5].
- Study the Pseudo-Smarandache-Pseudo-Cullen primes sequence, that is the primes of the form $n \cdot 2^{Z(n)} - 1$ where Z(n) is the Pseudo Smarandache function [6].
- Study the Smarandache-Pseudo-Cullen-Squarefree primes sequence, that is the primes of the form $n \cdot 2^{Zw(n)} - 1$ where Zw(n) is the Pseudo Smarandache Squarefree function
- For any of previous number-theoretic functions let a(n) be the Smarandache-Cullen primes and b(n) the Smarandache-Pseudo-Cullen primes. Evaluate the general continued fraction [2] for a(n) and b(n).

39) Smarandache generalized automorphic sequence

Let a(n) be any sequence with $n \geq 1$ and let's suppose that a(n)=m Then a(n) belongs to the sequence if and only if m ends in n.

- Study for example the Smarandache Automorphic Fibonacci sequence.
- Is this sequence finite?

40) Smarandache generalized doubly automorphic sequence

Let a(n) be any sequence with $n \geq 1$ and let suppose that a(n)=m
Then a(n) belongs to the sequence if and only if m ends in n and n itself belongs to the sequence.

- Study the Smarandache Doubly Automorphic Fibonacci sequence.
- Is that sequence finite?

41) Gilda's Numbers

0, 29, 49, 78, 110, 152, 220, 314, 330, 364, 440, 550, 628, 660, 683, 770, 880, 990, 997, 2207, 5346, 10891, 19075, 22125, 22502, 37396, 44627, 45381, 67883, 91893

If a Fibonacci sequence is formed with first term equal to the absolute value between decimal digits in n and second term equal to the sum of the decimal digits in n, then n itself occurs as a term in the sequence. Let be $n = x_1 x_2 \ldots x_i$ in base 10. Then $F(0) = |x_1 - x_2 \ldots x_i|$, $F(1) = x_1 + x_2 + \ldots x_i$, $F(k) = F(k-1) + F(k-2)$.
If $F(k) = n$ then n belongs to the sequence.

- Is that sequence infinite?

Conjecture: Yes

- Evaluate the continued fraction and continued radical for that sequence.
- Is the series $\sum_{n=1}^{\infty} \frac{1}{a(n)}$ convergent? If yes find the limit.

42) Gilda primes

29, 683, 997, 2207, 10891, 67883

Gilda's numbers that are primes.

- Is that sequence infinite?
- Is there any Gilda's palindromic prime?

43) Smarandache consecutive-deconstructive sequence

1, 23, 456, 78910, 1112131415, 161718192021, 22232425262728, 3031323334353637, ...

Start with natural numbers and then group them as showed below:

$1,\ \overline{2\ \ 3},\ \overline{4\ \ 5\ \ 6},\ \overline{7\ \ 8\ \ 9\ \ 10},\ \overline{11\ \ 12\ \ 13\ \ 14\ \ 15},\ \overline{16\ \ldots\ldots}$

- How many terms are primes?
- Find an expression for a(n) in terms of n.

44) Smarandache Fibo-deconstructive sequence

1, 12, 358, 13213455, 89144233377610, ...

Start with the Fibonacci numbers and then group them as showed below:

$1,\ \overline{1\ \ 2},\ \overline{3\ \ 5\ \ 8},\ \overline{13\ \ 21\ \ 34\ \ 55},\ \overline{89\ \ 144\ \ 233\ \ 377\ \ 610},\ \overline{987\ \ldots\ldots}$

- How many terms are primes?
- How many terms are Fibonacci numbers?
- Is there any perfect square among these numbers?

45) Smarandache generalized deconstructive sequence

Let a(n) be any sequence with $n \geq 1$. Then the sequence is formed by grouping the terms :

$a(1),\ \overline{a(2)\ a(3)},\ \overline{a(4)\ a(5)\ a(6)},\ \overline{a(7)\ a(8)a(9)a(10)},\ \overline{a(11)a(12)\ \ldots..}$

46) Smarandache consecutive-deconstructive sum sequence

1, 5, 15, 34, 65, 111, 175, 260, 369, 505, 671, 870, 1105, 1379, 1695, 2056, 2465, 2925, 3439, 4010, 4641, 5335, 6095, 6924, 7825, 8801, 9855, 10990, 12209, 13515, 14911, 16400, 17985, 19669, 21455, 23346

Sum of digits of Smarandache consecutive-deconstructive sequence terms.

- Note that the n-th term is given by: $\frac{n(n^2+1)}{2}$
- Evaluate the ratio $\frac{a(n)}{a(n+1)}$ where a(n) is the n-th term of the sequence.
- Evaluate $\lim_{n\to\infty} \frac{A-}{A+}$ where A- indicates how many terms are odd and A+ how many terms are even in the previous sequence.

47) Francesca-Carlotta's numbers

16, 21, 25, 50, 66, 102, 115, 154, 193, 291, 471, 573, 675, 777, 879, 2372, 4770, 3668, 6867, 22502, 22790, 32084, 41666, 46457

If a Fibonacci sequence is formed with first term equal to the number of digits in n and the second term equal to the sum of the decimal digits in n, then n itself occurs as a term in the sequence after the first two terms.

- Example: 16 belongs to the sequence, because the sequence 2,7,9,16,25,... contains 16.
- How many terms are primes?
- Is that sequence finite?
- Find an expression for a(n) in terms of n.
- How many terms are perfect square?

48) Smarandache circular generalized sequence

Let a(n) be any sequence with $n \geq 1$. The Smarandache circular generalized sequence is given by all the terms a(n) that still belong to the initial sequence on any cyclic rotation of their digits.

For example let's consider the sequence a(n) of even numbers:

0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, ...

The Smarandache circular even sequence is:

0, 2, 4, 6, 8, 20, 22, 24, 26, 28,....

because each term remain even for any cyclic rotation of its digits.
In the same way we can build the Smarandache circular prime sequence.
If n contains the digits 0, 2, 4, 6, 8 or 5 we can rotate them to the last place so that n is not a circular prime (we shall assume n contains at least 2 digits).
Thus any circular prime >9 can consist only of the digits 1, 3, 7, 9. One example of such a number is 9311 since :

1193, 3119, 9311 and 1931

are all primes. We call 1193 a primeval circular prime since it is the smallest number in this set of primes.
The following sequence of primeval circular primes:

2, 3, 5, 7, 11, 13, 17, 37, 79, 113, 197, 199, 337, 1193, 3779, 11939,19937, 193939,
199933, R19, R23, R317, R1031

where Rn indicates the repunit consisting of n ‘1s’, are all the primeval circular numbers known up to now.

- Is this sequence finite?

Here are some heuristics which suggest that the list is finite if we exclude the repunit primes. Consider numbers n with d digits, i.e. $10^{d-1} \leq n \leq 10-1$ (*)

The Prime Number Theorem [4] states (roughly) that the probability of a randomly chosen number of the size of n being prime is about:

$$p(n) \approx \frac{1}{\ln(n)} \approx \frac{1}{d \cdot \ln(10)}$$

The number of d digits integers n in the range (*) which consist only of the digits 1, 3, 7, or 9 is 4 so we might expect roughly

$$4^d \cdot \frac{1}{d \cdot \ln(10)}$$

of these to be prime. In fact this estimate has to be increased since such numbers are not chosen at random; they are specifically chosen not to be divisible by 2 or 5 so the expected number has to be multiplied by 2/1*5/4=5/2 to give the expected number :

$$\frac{5}{2} \cdot \frac{4^d}{d \cdot \ln(10)} = 1.086.... \cdot \frac{4^d}{d}$$

Any d-digits circular prime which is not a repunit must generate d distinct numbers by cycling. The probability that these are all primes is roughly

$$\left(\frac{5}{2}\right)^d \cdot \left(\frac{1}{d \cdot \ln(10)}\right)^d$$

and so we expect roughly:

$$4^d \cdot \left(\frac{5}{2}\right)^d \cdot \left(\frac{1}{d \cdot \ln(10)}\right)^d = \left(\frac{10}{d \cdot \ln(10)}\right)^d$$

d-digits circular primes. Since:

$$\sum_{d=1}^{\infty} \left(\frac{10}{d \cdot \ln(10)}\right)^d$$

converges we should only expect a finite number of such circular primes. Here is a table comparing the actual number of circular primes with the estimate:

d	Estimated	Actual

1	4.34294	4
2	4.71529	4
3	3.03381	4
4	1.38962	2
5	0.49439	2
6	0.14381	2
7	0.03538	0
8	0.00754	0
9	0.00141	0
10	0.00023	0
11	0.000036	0
12	0.000005	0

- Study the Smarandache circular Fibonacci sequence (1, 1, 2, 3, 5, 8, 55........)
- Is that sequence finite?
- Study the Smarandache circular Lucas sequence.
- Study the Smarandache circular Triangular sequence, namely the circular sequence obtained starting with the Triangular numbers.

49) Smarandache pseudo 2-expression

How many primes are in the following Smarandache-type expressions?

$$m \cdot n^m + n \cdot m^n + 1 \text{ for n and m=1, 2, 3, 4, 5....}$$

Conjecture: An infinte number.

$$n \cdot m^n + (n-1) \text{ where n and m=1, 2, 3, 4, 5 .....}$$

Conjecture: An infinite number

50) Conjecture on odd numbers

Every odd number n>1 can be written as:

$$n = 2 \cdot p + q$$

where p and q are two odd primes or 1.

The conjecture has been tested for all odd numbers $< 10^6$

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Chapter IV

An introduction to the Smarandache Double factorial function

In [1], [2] and [3] the Smarandache Double factorial function is defined as:

Sdf(n) is the smallest number such that Sdf(n)!! is divisible by n, where the double factorial is given by [4]:

m!! = 1x3x5x...m, if m is odd;
m!! = 2x4x6x...m, if m is even.

In this chapter we will study this function and several examples, theorems, conjectures and problems will be presented. The behaviour of this function is similar to the other Smarandache functions introduced in the chapter I.
In the table below the first 100 values of fucntion Sdf(n) are given:

n	Sdf(n)	n	Sdf(n)	n	Sdf(n)	n	Sdf(n)	n	Sdf(n)
1	1	21	7	41	41	61	61	81	15
2	2	22	22	42	14	62	62	82	82
3	3	23	23	43	43	63	9	83	83
4	4	24	6	44	22	64	8	84	14
5	5	25	15	45	9	65	13	85	17
6	6	26	26	46	46	66	22	86	86
7	7	27	9	47	47	67	67	87	29
8	4	28	14	48	6	68	34	88	22
9	9	29	29	49	21	69	23	89	89
10	10	30	10	50	20	70	14	90	12
11	11	31	31	51	17	71	71	91	13
12	6	32	8	52	26	72	12	92	46
13	13	33	11	53	53	73	73	93	31
14	14	34	34	54	18	74	74	94	94
15	5	35	7	55	11	75	15	95	19
16	6	36	12	56	14	76	38	96	8
17	17	37	37	57	19	77	11	97	97
18	12	38	38	58	58	78	26	98	28
19	19	39	13	59	59	79	79	99	11
20	10	40	10	60	10	80	10	100	20

According to the experimental data the following two conjectures can be formulated:

Conjecture 4.1 The series $\sum_{n=1}^{\infty} Sdf(n)$ is asymptotically equal to $a \cdot n^b$ where a and b are close to 0.8834.. and 1.759.. respectively.

Conjecture 4.2 The series $\sum_{n=1}^{\infty} \frac{1}{Sdf(n)}$ is asymptotically equal to $a \cdot n^b$ where a and b are close to 0.9411.. and 0.49.. respectively.

Let's start now with the proof of some theorems.

Theorem 4.1. Sdf(p)=p where p is any prime number.

Proof. For p=2, of course Sdf(2)=2. For p odd instead observes that only for m=p the factorial of first m odd integers is a multiple of p, that is $1 \cdot 3 \cdot 5 \cdot 7 \ldots \ldots p = (p-2)!! \cdot p$.

Theorem 4.2. For any squarefree even number n,

$$Sdf(n) = 2 \cdot \max\{p_1, p_2, p_3, \ldots \ldots p_k\}$$

where $p_1, p_2, p_3, \ldots \ldots p_k$ are the prime factors of n.

Proof. Without loss of generality let's suppose that $n = p_1, p_2, p_3$ where $p_3 > p_2 > p_1$ and $p_1 = 2$. Given that the factorial of even integers must be a multiple of n of course the smallest integer m such that $2 \cdot 4 \cdot 6 \ldots \cdot m$ is divisible by n is $2 \cdot p_3$. Infact for $m = 2 \cdot p_3$ we have :

$$2 \cdot 4 \cdot 6 \ldots 2 \cdot p_2 \ldots \cdot 2 \cdot p_3 = (2 \cdot p_2 \cdot p_3) \cdot (4 \cdot 6 \ldots 2 \ldots \cdot 2) = k \cdot (2 \cdot p_2 \cdot p_3) \ \ where\ k \in N$$

Theorem 4.3. For any squarefree composite odd number n, $Sdf(n) = \max\{p_1, p_2, \ldots p_k\}$ where $p_1, p_2, \ldots p_k$ are the prime factors of n.

Proof. Without loss of generality let suppose that $n = p_1 \cdot p_2$ where p_1 and p_2 are two distinct primes and $p_2 > p_1$. Of course the factorial of odd integers up to p_2 is a multiple of n because being $p_1 < p_2$ the factorial will contain the product $p_1 \cdot p_2$ and therefore n: $1 \cdot 3 \cdot 5 \ldots \cdot p_1 \ldots \cdot p_2$

Theorem 4.4. $\sum_{n=1}^{\infty} \frac{1}{Sdf(n)}$ diverges.

Proof. This theorem is a direct consequence of the divergence of sum $\sum_{p} \frac{1}{p}$ where p is any prime number.

In fact $\sum_{k=1}^{\infty} \frac{1}{Sdf(k)} > \sum_{p=2}^{\infty} \frac{1}{p}$ according to the theorem 4.1 and this proves the theorem.

Theorem 4.5 The Sdf(n) function is not additive that is $Sdf(n+m) \neq Sdf(n) + Sdf(m) \quad for \;\; (n,m) = 1$.

Proof. In fact for example $Sdf(2+15) \neq Sdf(2) + Sdf(15)$.

Theorem 4.6 The Sdf(n) function is not multiplicative, that is $Sdf(n \cdot m) \neq Sdf(n) \cdot Sdf(m) \quad for \;\; (n,m) = 1$.

Proof. In fact for example $Sdf(3 \cdot 4) \neq Sdf(3) \cdot Sdf(4)$.

Theorem 4.7 $Sdf(n) \leq n$

Proof. If n is a squarefree number then based on theorems 4.1, 4.2 and 4.3 $Sdf(n) \leq n$. Let's now consider the case when n is not a squarefree number. Of course the maximum value of the Sdf(n) function cannot be larger than n because when we arrive in the factorial to n for shure it is a multiple of n.

Theorem 4.8 $\sum_{n=1}^{\infty} \frac{Sdf(n)}{n}$ diverges.

Proof. In fact $\sum_{k=1}^{\infty} \frac{Sdf(k)}{k} > \sum_{p=2}^{\infty} \frac{Sdf(p)}{p}$ where p is any prime number and of course $\sum_{p} \frac{Sdf(p)}{p}$ diverges because the number of primes is infinite [5] and Sdf(p)=p.

Theorem 4.9 $Sdf(n) \geq 1 \ \ for \ \ n \geq 1$

Proof. This theorem is a direct consequence of the Sdf(n) function definition. In fact for n=1, the smallest m such that 1 divide Sdf(1) is trivially 1. For $n \neq 1$, m must be greater than 1 becuase the factorial of 1 cannot be a multiple of n.

Theorem 4.10 $0 < \frac{Sdf(n)}{n} \leq 1$ for $n \geq 1$

Poof. The theorem is a direct consequence of theorem 4.7 and 4.9.

Theorem 4. 11 $Sdf(p_k\#) = 2 \cdot p_k$ where $p_k\#$ is the product of first k primes (primorial) [4].

Proof. The theorem is a direct consequence of theorem 4.2.

Theorem 4.12 The equation $\frac{Sdf(n)}{n} = 1$ has an infinite number of solutions.

Proof. The theorem is a direct consequence of theorem 4.1 and the well-known fact that there is an infinite number of prime numbers [5].

Theorem 4.13 The even (odd respectively) numbers are invariant under the application of Sdf function, namely Sdf(even)=even and Sdf(odd)=odd

Proof. Of course this theorem is a direct consequence of the Sdf(n) function definition.

Theorem 4.14 The diophantine equation $Sdf(n) = Sdf(n+1)$ doesn't admit solutions.

Proof. In fact according to the previous theorem if n is even (odd respectively) then Sdf(n) also is even (odd respectively). Therefore the equation Sdf(n)=Sdf(n+1) can not be satisfied because Sdf(n) that is even should be equal to Sdf(n+1) that instead is odd.

Conjecture 4.3 The function $\frac{Sdf(n)}{n}$ is not distributed uniformly in the interval]0,1].

Conjecture 4.4 For any arbitrary real number $\boldsymbol{e} > 0$, there is some number $n \geq 1$ such that $\frac{Sdf(n)}{n} < \boldsymbol{e}$

Let's now start with some problems related to the Sdf(n) function.

Problem 1. *Use the notation FSdf(n)=m to denote, as already done for the Zt(n) and Zw(n) functions, that m is the number of different integers k such that Zw(k)=n.*

Example FSdf(1)=1 since Sdf(1)=1 and there are no other numbers n such that Sdf(n)=1

Study the function Fsdf(n).

Evaluate $\lim_{m\to\infty} \frac{\sum_{k=1}^{m} \frac{FSdf(k)}{k}}{m}$

Problem 2*.* *Is the difference |Sdf(n+1)-Sdf(n)| bounded or unbounded?*

Problem 3. *Find the solutions of the equations:* $\frac{Sdf(n+1)}{Sdf(n)} = k$ $\frac{Sdf(n)}{Sdf(n+1)} = k$ *where k is any positive integer and n>1 for the first equation.*

Conjecture 4.5 The previous equations don't admits solutions.

Problem 4. *Analyze the iteration of Sdf(n) for all values of n. For iteration we intend the repeated application of Sdf(n). For example the k-th iteration of Sdf(n) is:*

$$Sdf^{k}(n) = Sdf(Sdf(\ldots\ldots(Sdf(n))\ldots) \quad \textit{where Sdf is repeated k times.}$$

For all values of n, will each iteration of Sdf(n) produces always a fixed point or a cycle?

Problem 5. *Find the smallest k such that between Sdf(n) and Sdf(k+n), for n>1, there is at least a prime.*

Problem 6. *Is the number 0.1232567491011.... where the sequence of digits is Sdf(n) for* $n \geq 1$ *an irrational or trascendental number? (We call this number the Pseudo-Smarandache-Double Factorial constant).*

Problem 7. *Is the Smarandache Euler-Mascheroni sum (see chapter II for definition) convergent for Sdf(n) numbers? If yes evaluate the convergence value.*

Problem 8. *Evaluate* $\sum_{k=1}^{\infty} (-1)^{k} \cdot Sdf(k)^{-1}$

Problem 9. *Evaluate* $\prod_{n=1}^{\infty} \frac{1}{Sdf(n)}$

Problem 10. *Evaluate* $\lim_{k\to\infty}\frac{Sdf(k)}{\boldsymbol{q}(k)}$ *where* $\boldsymbol{q}(k)=\sum_{n\le k}\ln(Sdf(n))$

Problem 11. *Are there m, n, k non-null positive integers for which* $Sdf(n\cdot m)=m^{k}\cdot Sdf(n)$ *?*

Problem 12. *Are there integers k>1 and n>1 such that* $(Sdf(n))^{k}=k\cdot Sdf(n\cdot k)$ *?*

Problem 13. *Solve the problems from 1 up to 6 already formulated for the Zw(n) function also for the Sdf(n) function.*

Problem 14. *Find all the solution of the equation Sdf(n)!=Sdf(n!)*

Problem 15. *Find all the solutions of the equation* $Sdf(n^{k})=k\cdot Sdf(n)$ *for k>1and n>1.*

Problem 16. *Find all the solutions of the equation* $Sdf(n^{k})=n\cdot Sdf(k)$ *for k>1.*

Problem 17. *Find all the solutions of the equation* $Sdf(n^{k})=n^{m}\cdot Sdf(m)$ *where k>1 and n, m >0.*

Problem 18. *For the first values of the Sdf(n) function the following inequality is true:*

$$\frac{n}{Sdf(n)}\le\frac{1}{8}\cdot n+2\quad \textit{for}\ 1\le n\le 1000$$

Is this still true for n>1000 ?

Problem 19. *For the first values of the Sdf(n) function the following inequality is true:*

$$\frac{Sdf(n)}{n} \leq \frac{1}{n^{0.73}} \quad for \ \ 1 \leq n \leq 1000$$

Is this still true for all values of n>1000 ?

Problem 20. *For the first values of the Sdf(n) function the following inequality hold:*

$$\frac{1}{n} + \frac{1}{Sdf(n)} < n^{-\frac{1}{4}} \quad for \ \ 2 < n \leq 1000$$

Is this still true for n>1000 ?

Problem 21. *For the first values of the Sdf(n) function the following inequality holds:*

$$\frac{1}{n \cdot Sdf(n)} < n^{-\frac{5}{4}} \quad for \ \ 1 \leq n \leq 1000$$

Is this inequality still true for n>1000 ?

Problem 22. *Study the convergence of the Smarandache Double factorial harmonic series:*

$$\sum_{n=1}^{\infty} \frac{1}{Sdf^{a}(n)} \quad where \ \ a > 0 \ \ and \ \ a \in R$$

Problem 23. *Study the convergence of the series:*

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{Sdf(x_n)}$$

where x_n *is any increasing sequence such that* $\lim\limits_{n\to\infty} x_n = \infty$

Problem 24. *Evaluate*

$$\lim_{n\to\infty} \frac{\sum\limits_{k=2}^{n} \frac{\ln(Sdf(k))}{\ln(k)}}{n}$$

Is this limit convergent to some known mathematical constant?

Problem 25. *Solve the functional equation:*

$$Sdf(n)^r + Sdf(n)^{r-1} + \cdots \cdots Sdf(n) = n$$

where r is an integer ≥ 2 .
Wath about the functional equation:

$$Sdf(n)^r + Sdf(n)^{r-1} + \cdots \cdots Sdf(n) = k \cdot n$$

where r and k are two integers ≥ 2 .

Problem 26. *Is there any relationship between* $Sdf\left(\prod\limits_{k=1}^{m} m_k\right)$ *and* $\sum\limits_{k=1}^{m} Sdf(m_k)$?

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Chapter V

On some Smarandache conjectures and unsolved problems

In this chapter some Smarandache conjectures and open questions will be analysed. The first three conjectures are related to prime numbers and formulated by F. Smarandache in [1].

1) First Smarandache conjecture on primes

The equation:

$$B_n(x) = p_{n+1}^x - p_n^x = 1 \ ,$$

where p_n is the n-th prime, has a unique solution between 0.5 and 1;

- the maximum solution occurs for n = 1, i.e.

$$3^x - 2^x = 1 \qquad when \quad x = 1;$$

- the minimum solution occurs for n = 31, i.e.

$$127^x - 113^x = 1 \qquad when \quad x = 0.567148\ldots = a_0$$

First of all observe that the function $B_n(x)$ which graph is reported in the fig. 5.1 for some values of n, is an increasing function for $x > 0$ and then it admits a unique solution for $0.5 \le x \le 1$.

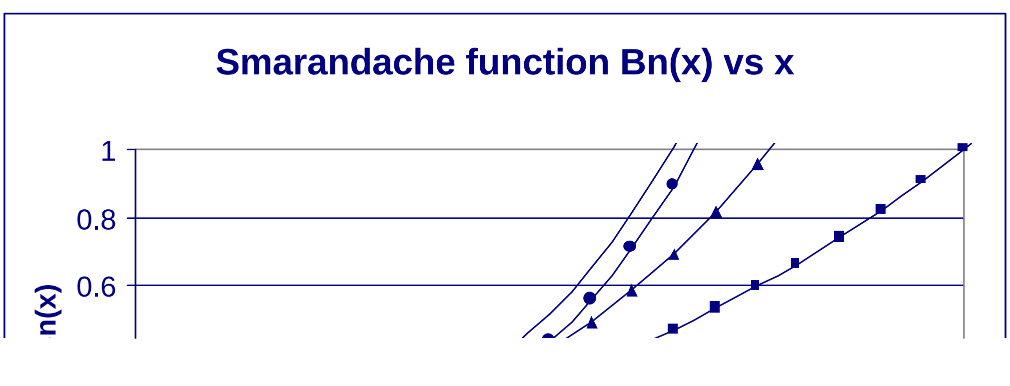

Fig. 5.1

In fact the derivate of $B_n(x)$ function is given by:

$$\frac{d}{dx}B_n(x) = p_{n+1}^x \cdot \ln(p_{n+1}) - p_n^x \cdot \ln(p_n)$$

and then since $p_{n+1} > p_n$ we have:

$$\ln(p_{n+1}) > \ln(p_n) \quad and \quad p_{n+1}^x > p_n^x \quad for\ x > 0$$

This implies that $\frac{d}{dx}B_n(x) > 0 \quad for \quad x > 0 \;\; and \;\; n > 0$.

Being the $B_n(x)$ an increasing fucntion, the Smarandache conjecture is equivalent to:

$$B_n^0 = p_{n+1}^{a_0} - p_n^{a_0} \leq 1$$

that is, the intersection of $B_n(x)$ function with $x = a_0$ line is always lower or equal to 1. Then an Ubasic program has been written to test the new version of Smarandache conjecture for all primes lower than 2^{27} . In this range the conjecture is true. Moreover we have created an histogram for the intersection values of $B_n(x)$ with $x = a_0$:

Counts	Interval
7600437	[0, 0.1]
2640	]0.1, 0.2]
318	]0.2, 0.3]
96	]0.3, 0.4]
36	]0.4, 0.5]
9	]0.5, 0.6]
10	]0.6, 0.7]
2	]0.7, 0.8]

This means for example that the function $B_n(x)$ intersects the axis $x = a_0$, 318 times in the interval]0.2, 0.3] for all n such that $p_n < 2^{27}$.
In the fig. 5.2 the graph of normalized histrogram is reported (black dots).
According to the experimental data an interpolating function has been estimated (continous curve):

$$B_n^0 = 8 \cdot 10^{-8} \cdot \frac{1}{n^{6.2419}}$$

with a good R^2 value (97%).

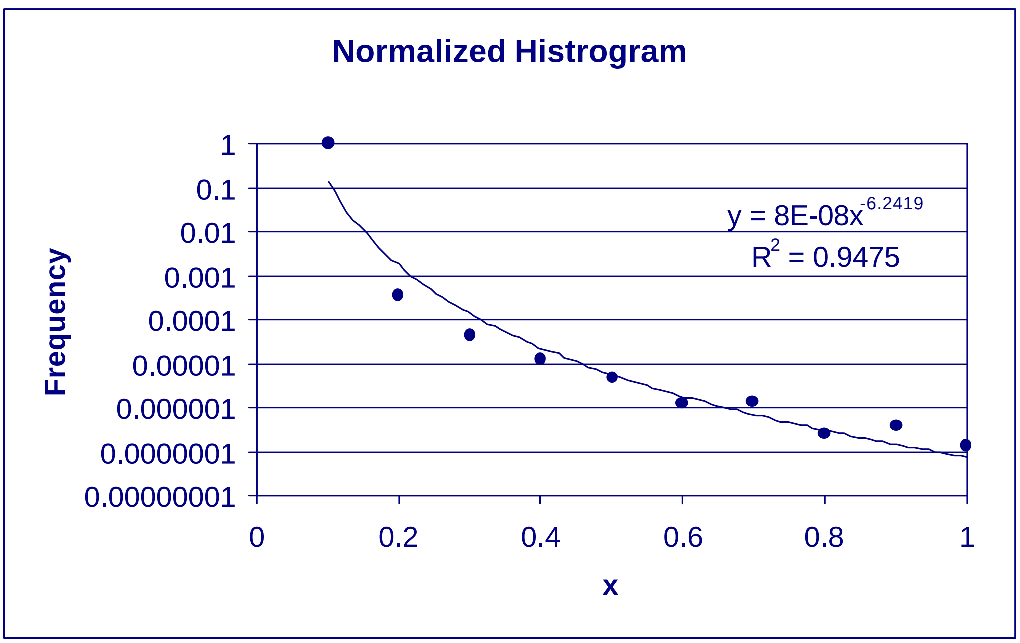

Fig. 5.2

Assuming this function as empirical probability density function we can evaluate the probability that $B_n^0 > 1$ and then that the Smarandache coinjecture is false.

By definition of probability we have:

$$P(B_n^0 > 1) = \frac{\int_1^\infty B_n^0\, dn}{\int_c^\infty B_n^0\, dn} \approx 6.99 \cdot 10^{-19}$$

where c=3.44E-4 is the lower limit of B_n^0 found with our computer search. Based on those experimental data there is a strong evidence that the Smarandache conjecture on primes is true.

2) Second Smarandache conjecture on primes.

$$B_n(x) = p_{n+1}^x - p_n^x \quad < 1$$

where $x < a_0$. Here p_n is the n-th prime number.
This conjecture is a direct consequence of conjecture number 1 analysed before. In fact being $B_n(x)$ an increasing function if:

$$B_n^0 = p_{n+1}^{a_0} - p_n^{a_0} \le 1$$

is verified then for $x < a_0$ we have no intersections of the $B_n(x)$ function with the line $B_n(x) = 1$, and then $B_n(x)$ is always lower than 1.

3) Third Smarandache conjecture on primes.

$$C_n(k) = p_{n+1}^{\frac{1}{k}} - p_n^{\frac{1}{k}} < \frac{2}{k} \qquad \textit{for } \; k \ge 2 \;\; \textit{and} \;\; p_n \;\; \textit{the } n-th \;\; \textit{prime number}$$

This conjecture has been verified for prime numbers up to 2^{25} and $2 \le k \le 10$ by the author [2]. Moreover a heuristic that highlight the validity of conjecture out of range analysed was given too.
At the end of the paper the author reformulated the Smarandache conjecture in the following one:

Smarandache-Russo conjecture

$$C_n(k) \le \frac{2}{k^{2 \cdot a_0}} \qquad for \quad k \ge 2$$

where a_0 is the Smarandache constant $a_0 = 0.567148...$ (see [1]).
So in this case for example the Andrica conjecture (namely the Smarandache conjecture for k=2) becomes:

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 0.91111....$$

Thanks to a program written with Ubasic software the conjecture has been verified to be true for all primes $p_n < 2^{25}$ and $2 \le k \le 15$.
In the following table the results of the computer search are reported.

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Max_C(n,k)	0.6708	0.3110	0.1945	0.1396	0.1082	0.0885	0.0756	0.0659	0.0584	0.0525	0.0476	0.0436	0.0401	0.0372
2/k^(2a0)	0.4150	0.1654	0.0861	0.0519	0.0343	0.0242	0.0178	0.0136	0.0107	0.0086	0.0071	0.0060	0.0050	0.0043
delta	0.2402	0.2641	0.2204	0.1826	0.1538	0.1314	0.1134	0.0994	0.0883	0.0792	0.0717	0.0654	0.0600	0.0554

Max_C(n,k) is the largest value of the Smarandache function $C_n(k)$ for $2 \le k \le 15$ and $p_n < 2^{25}$ and delta is the difference between $\frac{2}{k^{2 \cdot a_0}}$ and Max_C(n,k).
Let's now analyse the behaviour of the delta function versus the k parameter. As highlighted in the following graph (fig. 5.3),

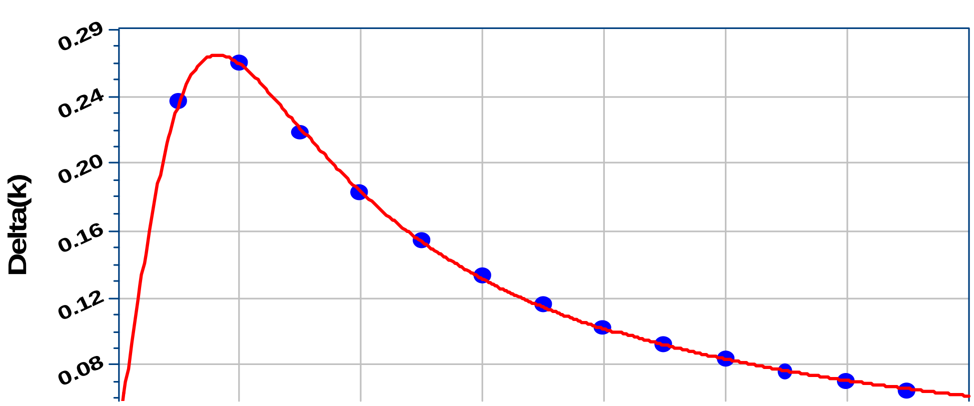

Fig. 5.3

an interpolating function with good R^2 (0.999) has been estimated:

$$Delta(k) = \frac{a + b \cdot k}{1 + c \cdot k + d \cdot k^2}$$

where: $a = 0.1525...$, $b = 0.17771..$, $c = -0.5344....$, $d = 0.2271...$

Since the Smarandache function decrease asymptotically as n increases it is likely that the estimated maximum is valid also for $p_n > 2^{25}$. If this is the case then the interpolating function found reinforce the Smarandache-Russo conjecture being:

$$Delta(k) \to 0 \quad for \;\; k \to \infty$$

Let's now analyse some Smarandache conjectures that are a generalization of Goldbach conjecture.

4) Smarandache generalization of Goldbach conjectures

C. Goldbach (1690-1764) was a German mathematician who became professor of mathematics in 1725 in St. Petersburg, Russia. In a letter to Euler on June 7, 1742, He speculated that every even number is the sum of three primes.

Goldbach in his letter was assuming that 1 was a prime number. Since we now exclude it as a prime, the modern statements of Goldbach's conjectures are [5]:

Every even number equal or greater than 4 can be expressed as the sum of two primes, and every odd number equal or greater than 9 can be expressed as the sum of three primes.

The first part of this claim is called the Strong Goldbach Conjecture, and the second part is the Weak Goldbach Conjecture.

After all these years, the strong Goldbach conjecture is still not proven, even though virtually all mathematicians believe it is true.
Goldbach's weak conjecture has been proven, almost!
In 1937, I.M. Vonogradov proved that there exist some number N such that all odd numbers that are larger than N can be written as the sum of three primes. This reduce the problem to finding this number N, and then testing all odd numbers up to N to verify that they, too, can be written as the sum of three primes.

How big is N? One of the first estimates of its size was approximately [6]:

$$10^{6846168}$$

But this is a rather large number; to test all odd numbers up to this limit would take more time and computer power than we have. Recent work has improved the estimate of N. In 1989 J.R. Chen and T. Wang computed N to be approximately [7]:

$$10^{43000}$$

This new value for N is much smaller than the previous one, and suggests that some day soon we will be able to test all odd numbers up to this limit to see if they can be written as the sum of three primes.
Anyway assuming the truth of the generalized Riemann hypothesis [5], the number N has been reduced to 10^{20} by Zinoviev [9], Saouter [10] and Deshouillers. Effinger, te Riele and Zinoviev[11] have now successfully reduced N to 5.
Therefore the weak Goldbach conjecture is true, subject to the truth of the generalized Riemann hypothesis.
Let's now analyse the generalizations of Goldbach conjectures reported in [3] and [4]; six different conjectures for odd numbers and four conjectures for even numbers have been formulated. We will consider only the conjectures 1, 4 and 5 for the odd numbers and the conjectures 1, 2 and 3 for the even ones.

4.1 First Smarandache Goldbach conjecture on even numbers.

Every even integer n can be written as the difference of two odd primes, that is $n = p - q$ *with p and q two primes.*

This conjecture is equivalent to:

For each even integer n, we can find a prime q such that the sum of n and q is itself a prime p.

A program in Ubasic language to check this conjecture has been written.
The result of this check has been that the first Smarandache Goldbach conjecture is true for all even integers equal or smaller than 2^{29}.
The list of Ubasic program follows.

```
   1  ' ****************************************************
   2  '                    Smarandache Goldbach conjecture
   3  '          on even numbers: n=p-q with p and q two primes
   4  '                         by Felice Russo Oct. 1999
   5  ' ****************************************************
  10  cls
  20  for N=2 to 2^28 step 2
  22  W=3
  25  locate 10,10:print N
  30  for Q=W to 10^9
  40  gosub *Pspr(Q)
  50  if Pass=0 then goto 70
  60  cancel for:goto 80
  70  next
  75  print N,"The Smarandache conjecture is not true up to 10^9 for q=";Q
  80  Sum=N+Q
  90  gosub *Pspr(Sum)
 100  if Pass=1 then goto 120
 110  W=Q+1:goto 30
 120  next
 130  print "The Smarandache conjecture has been verified up to:";N-2
 140  end
1000  ' ****************************************
1010  '       Strong Pseudoprime Test Subroutine
1020  '             by Felice Russo 25/5/99
1030  ' ****************************************
1040  '
1050  ' The sub return the value of variable PASS.
1060  ' If pass is equal to 1 then N is a prime.
1070  '
1080  '
1090  *Pspr(N)
```

```
1100   local I,J,W,T,A,Test
1110   W=3:if N=2 then Pass=1:goto 1290
1120   if even(N)=1 or N=1 then Pass=0:goto 1290
1130   if prmdiv(N)=N then Pass=1:goto 1290
1140   if prmdiv(N)>0 and prmdiv(N)<N then Pass=0:goto 1290
1150   I=W
1160   if gcd(I,N)=1 then goto 1180
1170   W=I+1:goto 1150
1180   T=N-1:A=A+1
1190   while even(T)=1
1200   T=T\2:A=A+1
1210   wend
1220   Test=modpow(I,T,N)
1230   if Test=1 or Test=N-1 then Pass=1:goto 1290
1240   for J=1 to A-1
1250   Test=(Test*Test)@N
1260   if Test=N-1 then Pass=1:cancel for:goto 1290
1270   next
1280   Pass=0
1290   return
```

For each even integer n the program check if it is possible to find a prime q, generated by a subroutine (rows from 1000 to 1290) that tests the primality of an integer, such that the sum of n and q, sum=n+q (see rows 80 and 90) is again a prime.

If yes the program jumps to the next even integer. Of course we have checked only a little quantity of integers out of infinite number of them.

Anyway we can get some further information from experimental data about the validity of this conjeture.

In fact we can calculate the ratio q/n for the first 3000 values, for example, and then graphs this ratio versus n (see fig. 5.4).

Fig. 5.4

As we can see this ratio is a decreasing function of n; this means that for each n is very easy to find a prime q such that n+q is a prime. This heuristic well support the Smarandache-Goldbach conjecture.

4.2 Second Smarandache-Goldbach conjecture on even numbers.

Every even integer n can be expressed as a combination of four primes as follows:

$$n=p+q+r-t \text{ where } p, q, r, t \text{ are primes.}$$

For example: 2=3+3+3-7, 4=3+3+5-7, 6=3+5+5-7, 8=11+5+5-13 …….
Regarding this conjecture we can notice that since n is even and t is an odd prime their sum is an odd integer.
So the conjecture is equivalent to the weak Goldbach conjecture because we can always choose a prime t such that $n+t \geq 9$.

4.3 Third Smarandache-Goldbach conjecture on even numbers.

Every even integer n can be expressed as a combination of four primes as follows:

$$n=p+q-r-t \text{ where } p, q, r, t \text{ are primes.}$$

For example: 2=11+11-3-17, 4=11+13-3-17, 6=13+13-3-17, 8=11+17-7-13 ….

As before this conjecture is equivalent to the strong Goldbach conjecture because the sum of an even integer plus two odd primes is an even integer. But according to the Goldbach conjecture every even integer ≥ 4 can be expressed as the sum of two primes.

4.4 First Smarandache Goldbach conjecture on odd numbers.

Every odd integer n, can be written as the sum of two primes minus another prime:

$$n=p+q-r \quad \text{where } p, q, r \text{ are prime numbers.}$$

For example: 1=3+5-7,   3=5+5-7,   5=3+13-11, 7=11+13-17   9=5+7-3 ….

Since the sum of an odd integer plus an odd prime is an even integer this conjecture is equivalent to the strong Goldbach conjecture that states that every even integer ≥ 4 can be written as the sum of two prime numbers.
A little variant of this conjecture can be formulated requiring that all the three primes must be different.
For this purpose an Ubasic program has been written. The conjecture has been verified to be true for odd integers up to 2^{29}.
The algorithm is very simple. In fact for each odd integer n, we put r=3, p=3 and q equal to the largest primes smaller than n+r.
Then we check the sum of p and q. If it is greater than n+r then we decrease the variable q to the largest prime smaller than the previous one. On the contrary if the sum is smaller than n+r we increase the variable p to the next prime. This loop continues untill p is lower than q. If this is not the case then we increase the variable r to the next prime and we restart again the check on p and q. If the sum of n and r coincide with that of p and q the last check is on the three primes r, p and q that must be of course different. If this is not the case then we reject this solution and start again the check.

```
' ********************************************
'            First Smarandache-Goldbach conjecture
'                          on odd integers
'                     by Felice Russo Oct. 99
```

```
 5 ' *******************************************
10 cls:Lim=2^29
20 for N=1 to Lim step 2
30 S=3:W=3
40 locate 10,10:print N
50 r=S
60 gosub *Pspr(r)
70 if Pass=0 then goto 260
80 Sum1=N+r:L=0:H=Sum1-1
90 p=W
100 gosub *Pspr(p)
110 if Pass=1 and L=0 then goto 140
120 if Pass=1 and L=1 then goto 190
130 W=p+1:goto 90
140 q=H
150 gosub *Pspr(q)
160 if Pass=1 then goto 190
170 H=q-1:goto 140
190 Sum2=p+q
200 if p>=q then goto 260
210 if Sum2>Sum1 then H=q-1:goto 140
220 if Sum2<Sum1 then W=p+1:L=1:goto 90
230 if r=p or r=q and p<q then W=p+1:goto 90
240 if r=p or r=q and p>=q then goto 260
250 goto 270
260 S=r+1:if r>2^25 goto 290 else goto 50
270 next
280 cls:print "Conjecture verified up to";Lim:goto 300
290 cls:print "Conjecture not verified up to 2^25 for";N
300 end
310 ' **************************************
320 '        Strong Pseudoprime Test Subroutine
330 '               by Felice Russo 25/5/99
340 ' **************************************
350 '
360 ' The sub return the value of variable PASS.
370 ' If pass is equal to 1 then N is a prime.
380 '
390 '
400 *Pspr(N)
410 local I,J,W,T,A,Test
```

```
420  W=3:if N=2 then Pass=1:goto 600
430  if even(N)=1 or N=1 then Pass=0:goto 600
440  if prmdiv(N)=N then Pass=1:goto 600
450  if prmdiv(N)>0 and prmdiv(N)<N then Pass=0:goto 600
460  I=W
470  if gcd(I,N)=1 then goto 490
480  W=I+1:goto 460
490  T=N-1:A=A+1
500  while even(T)=1
510  T=T\2:A=A+1
520  wend
530  Test=modpow(I,T,N)
540  if Test=1 or Test=N-1 then Pass=1:goto 600
550  for J=1 to A-1
560  Test=(Test*Test)@N
570  if Test=N-1 then Pass=1:cancel for:goto 600
580  next
590  Pass=0
600  return
```

4.5 Fourth Smarandache Goldbach conjecture on odd numbers.

Every odd integer n can be expressed as a combination of five primes as follows:

$$n=p+q+r-t-u \quad \text{where } p, q, r, t, u \text{ are all prime numbers.}$$

For example: 1=3+7+17-13-13, 3=5+7+17-13-13, 5=7+7+17-13-13, 7=5+11+17-13-13

Also in this case the conjecture is equivalent to the weak Goldbach conjecture. In fact the sum of two odd primes plus an odd integer is alway an odd integer and according to the weak Goldbach conjecture it can be expressed as the sum of three primes.

Now we will analyse a conjecture about the wrong numbers introduced in Number Theory by F. Smarandache and reported for instance in [8] and then we will analyse a problem proposed by Castillo in [12].

5) Smarandache Wrong numbers

A number $n=\overline{a1a2a3...ak}$ of at least two digits, is said a Smarandache Wrong number if the sequence:

$$a_1, a_2, a_3, \ldots\ldots, a_k, b_{k+1}, b_{k+2}, \ldots\ldots$$

(where b_{k+i} is the product of the previous k terms, for any $i \geq 0$) contains n as its term [8].
Smarandache conjectured that there are no Smarandache Wrong numbers.
In order to check the validity of this conjecture up to some value N_0, an Ubasic program has been written.
N_0 has been choose equal to 2^{28}. For all integers $n \leq N_0$ the conjecture has been proven to be true. Moreover utilizing the experimental data obtained with the computer program a heuristic that reinforce the validity of conjecture can be given.
First of all let's define what we will call the Smarandache Wrongness of an integer n with at least two digits. For any integer n, by definition of Smarandache Wrong number we must create the sequence:

$$a_1, a_2, a_3, \ldots\ldots, a_k, b_{k+1}, b_{k+2}, \ldots\ldots$$

as reported above. Of course this sequence is stopped once a term b_{k+i} equal or greater than n is obtained.
Then for each integer n we can define two distance:

$$d_1 = |b_{k+i} - n| \qquad and \qquad d_2 = |b_{k+i-1} - n|$$

The Smarandache Wrongness of n is defined as $\min\{d_1, d_2\}$ that is the minimum value between d1 and d2 and indicate with W(n). Based on definition of W(n), if the Smarandache conjecture is false then for some n we should have W(n)=0.
Of course by definition of wrong number, W(n)=n if n contains any digit equal to zero and W(n)=n-1 if n is repunit (that is all the digits are 1). In the following analysis we will exclude this two species of integers. With the Ubasic program utilized to test the smarandache conjecture we have calculated the W(n) function for $12 \leq n \leq 3000$. The graph of W(n) versus n follows.

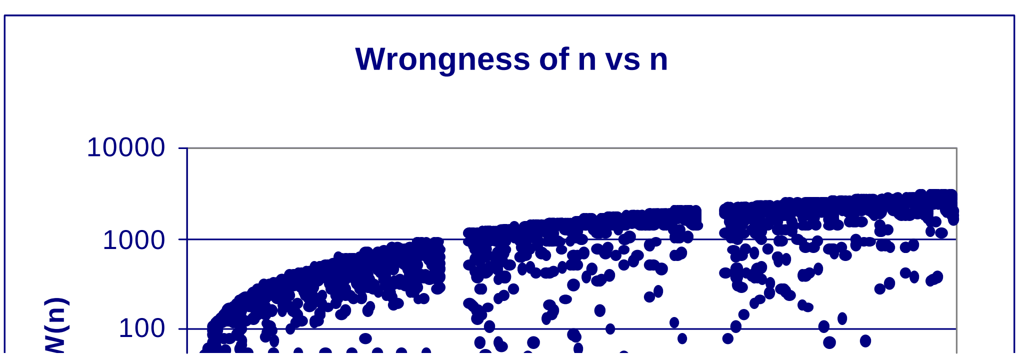

Fig. 5.5

As we can see W(n) in average increases linearly with n even though at a more close view (see fig. 5.6) a nice triangular pattern emerges with points scattered in the region between the x-axis and the triangles.
Anyway the average behaviour of W(n) function seems to support the validity of Smarandache conjecture.

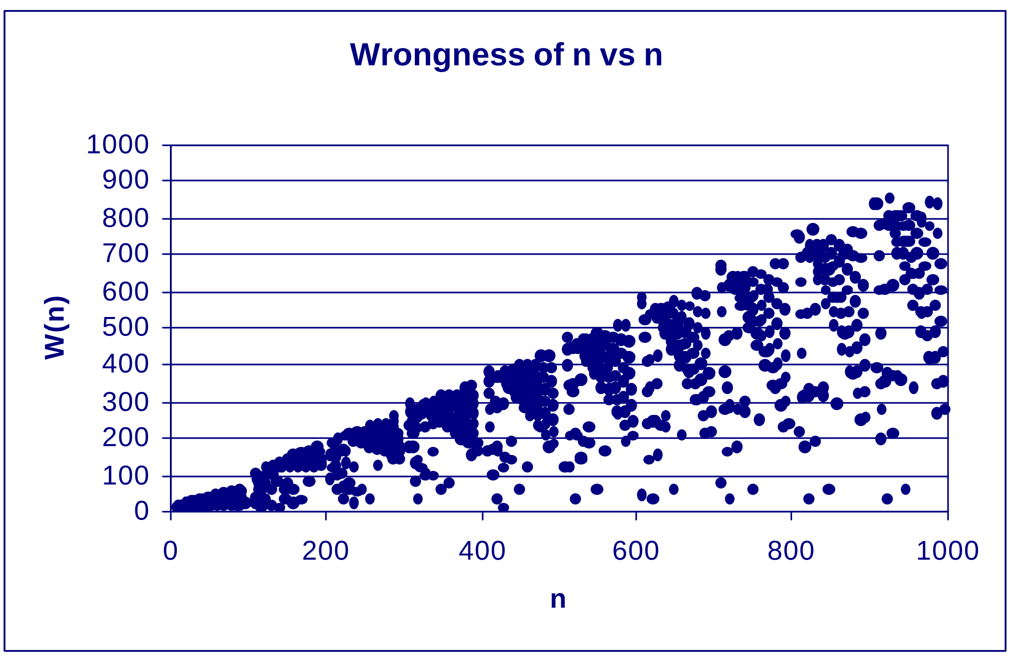

Fig. 5.6

Let's now divides the integers n in two family: those which W(n) function is smaller than 5 and those which W(n) function is greater than 5.
The integers with W(n) smaller than 5 will be called the Smarandache Weak Wrong numbers.

n	W(n)
12	4
13	4
14	2
23	5
31	4
143	1
431	1
1292	4
1761	3
2911	5
6148	4
11663	1
23326	2
314933	5
5242881	1

interv.	C_Ww(n)
10^2	5
10^3	2
10^4	4
10^5	2
10^6	1
10^7	1
10^8	0
2^{28}	0
2^{29}	0

Up to 2^{28} the sequence of weak wrong numbers is given by the following integers n:

Here W(n) is the Wrongness of n and C_Ww(n) is the number of the weak wrong numbers between 10 and 10^2, 10^2 and 10^3 and so on.
Once again the experimental data well support the Smarandache conjecture because the density of the weak wrong numbers seems goes rapidly to zero.

6) About a problem on continued fraction of Smarandache consecutive and reverse sequences.

In [12] J. Castillo introduced the notion of Smarandache simple continued fraction and Smarandache general continued fraction. As example he considered the application of this new concept to the two well-know Smarandache sequences:

Smarandache consecutive sequence

1, 12, 123, 1234, 12345, 123456, 1234567 ......

Smarandache reverse sequence

1, 21, 321, 4321, 54321, 654321, 7654321 ....

At the end of its article the following problem has been formulated:

Is the simple continued fraction of consecutive sequence convergent? If yes calculate the limit.

$$1+\cfrac{1}{12+\cfrac{1}{123+\cfrac{1}{1234+\cfrac{1}{12345+\cdots}}}}$$

Is the general continued fraction of consecutive and reverse sequences convergent? If yes calculate the limit.

$$1+\cfrac{1}{12+\cfrac{21}{123+\cfrac{321}{1234+\cfrac{4321}{12345+\cdots}}}}$$

Using the Ubasic software a program to calculate numerically the above continued fractions has been written. Here below the result of computation.

$$1+\cfrac{1}{12+\cfrac{1}{123+\cfrac{1}{1234+\cfrac{1}{12345+\ldots}}}}\approx 1.0833....$$

$$1+\cfrac{1}{12+\cfrac{21}{123+\cfrac{321}{1234+\cfrac{4321}{12345+\ldots}}}}\approx 1.0822....\approx K_e$$

where K_e is the Keane's constant (see [13])
Moreover for both the sequences the continued radical (see chapter II) and the Smarandache series [14] have been evaluated too.

$$\sqrt{1+\sqrt{12+\sqrt{123+\sqrt{1234+\ldots\ldots}}}}\approx 2.442.....\approx\frac{2}{7}\cdot\sin(\frac{\pi}{18})$$

$$\sqrt{1+\sqrt{21+\sqrt{321+\sqrt{4321+\ldots\ldots}}}}\approx 2.716...\approx\lim_{x\to\infty}(1+x)^{\frac{1}{x}}=e$$

$$\sum_{n=1}^{\infty}\frac{1}{a(n)}\approx 1.0924......\approx B$$

where a(n) is the Smarandache consecutive sequence and B the Brun's constant [15].

$$\sum_{n=1}^{\infty}\frac{1}{b(n)}\approx 1.051....$$

where b(n) is the Smarandache reverse sequence.

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